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The Navier-Stokes Equation with the kinematic and Vorticity boundary condition on Non-flat Boundaries

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Abstract

We study the initial-boundary value problem of the Navier-Stokes equations for incompressible fluids in a general domain in $\mathbb{R}^n$ with compact and smooth boundary, subject to the kinematic and vorticity boundary conditions on the non-flat boundary. Then we establish the well-posedness of the unsteady Stokes equations and employ the solution to reduce our initial-boundary value problem into an initial-boundary value problem with absolute boundary conditions.

Keywords Navier-Stokes equations, incompressible, vorticity boundary condition, kine-matic boundary condition, absolute boundary condition , non-flat boundary, vor-ticity; strong solutions, inviscid limit, slip boundary condition.

1. INTRODUCTION

The motion of an incompressible viscous fluid in $\mathbb{R}^n, n \geq 2$, is described by the Navier – stoke equation :

$$\partial_t u + u.\nabla u + \nabla p = \mu\Delta u, \quad \rightarrow (1.1)$$

$$\nabla.u \quad \rightarrow (1.2)$$

$$\frac{d}{dt}\|h\|_2^2 = -2\mu\int_\Omega\langle g, \nabla\times(\nabla\times g)\rangle dx$$

$$= -2\mu\int_\Omega\langle\nabla\times g, \nabla\times g\rangle dx - 2\mu\int_\Gamma\langle(\nabla\times g)^{\parallel}\times a, v\rangle ds$$

With initial data $u|_{t=0} = u_0(x), \rightarrow (1.3)$
Where $u(t,x) = (u^1, \dots\dots, u^n)(t,x)$ is the velocity field and $p(t,x)$ is the pressure that maintains the incompressibility of a fluid at (t,x). i.e., $divu$ =0 is the incompressibility condition . As a nonlinear system of a partial differential equation u and p are regarded as unknown functions, and the initial velocity field $u_0(x)$ sets the fluid in motion .

The constant $\mu > 0$ is the kinematic viscosity constant, $u.\nabla = 0$ denotes the covariant derivative along the flow trajectories , namely ,the directional derivative in the direction $u, \Delta u$ is the usual Laplacian on u and $\mu\Delta u$ represents the stress to the fluid . As usual, usual, we use $\nabla\cdot$=div to denote the divergence operator .

for inviscid flow , $\mu = 0$ and then equation are referred to as the Euler equation for incompressible fluid flow:

$$\partial_t u + u.\nabla u + \nabla p = \mu\Delta u, \rightarrow (1.4)$$

$$\nabla.u = 0 \quad \rightarrow (1.5)$$

When a fluid is confined in a bounded domain $\Omega \subset \mathbb{R}^n$ with non – empty boundary Γ, these equation must be supplied with proper boundary condition in order to be well- posed . For concreteness,the bounded domain $\Omega \subset \mathbb{R}^n$ is assumed to have a compact ,oriented, smooth surface boundary Γ .

In this paper ,we propose and study the initial- boundary value problem

$$\partial_t u + u.\nabla u + \nabla p = \mu\Delta u,$$

$$\nabla.u = 0$$

With initial data $u|_{t=0} = u_0(x)$,

With the following boundary condition on the non –flat boundary Γ:

$$u^{\perp}\big|_\Gamma = 0 \quad \text{(kinematic condition)} \quad \rightarrow (1.6)$$

$$\omega^{\parallel}\big|_\Gamma = a \quad \text{(vorticity condition)} \quad \rightarrow (1.7)$$

Where $\omega = \nabla \times u \in \mathbb{R}^n$ is the vorticity , the field $a = a(t,x)$ is defined on the non-flat

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boundary Γ, $u^{\perp} \in \mathbb{R}$ denotes the normal component and $u^{\parallel}$ the tangential part of $u \in \mathbb{R}^n$

1.1 Theorem

Let u and w be two vector fields on . then $\langle w.\nabla u, v\rangle = -\pi(u^{\parallel}, w^{\parallel})$

$$- H\langle w,v\rangle\langle u,v\rangle + \langle w,v\rangle(\nabla.u) + \langle w^{\parallel}, \nabla^{\Gamma}\langle u,v\rangle\rangle + \langle u^{\parallel}, \nabla^{\Gamma}\langle u,v\rangle\rangle - \nabla^{\Gamma}.(\langle w,v\rangle u^{\parallel})$$

In particular ,if $u^{\perp} = w^{\perp} = 0$ on Γ ,

Then , $\langle w.\nabla u, v\rangle = -\pi(u, w)$

Proof:

Given:

Let u and w be two vector fields on Ω

Toprove:

$$\langle w.\nabla u, v\rangle = -\pi(u^{\parallel}, w^{\parallel})\text{-}H\langle w,v\rangle\langle u,v\rangle\langle w,v\rangle(\nabla.u) + \langle w^{\parallel}, \nabla^{\Gamma}\langle u,v\rangle\rangle + \langle u^{\parallel}, \nabla^{\Gamma}\langle u,v\rangle\rangle - \nabla^{\Gamma}.\left(\langle w,v\rangle u^{\parallel}\right) \rightarrow (1.1.1)$$

$$\langle w.\nabla u, v\rangle = -\pi(u, w) \quad \rightarrow (1.1.2)$$

Since $(w.\nabla u)^i = w^j \left(e_j\left(u^i\right) + {\Gamma^i}_{jk} u^k\right)$

And then $\langle w.\nabla u, v\rangle = w^j \left(e_j\left(u^i\right) + {\Gamma^i}_{jk} u^k\right)$

$$\langle w.\nabla u, v\rangle = \langle w^{\parallel}, \nabla^{\Gamma}\langle u,v\rangle\rangle \ -\pi(u^{\parallel}, w^{\parallel}) + w^n(\nabla_n u^n) \rightarrow (1.1.3)$$

On the other hand, according to the Ricci equation,

$$\sum_{j=1}^{n-1} \nabla_j u^j = \sum_{j=1}^{n-1} {\nabla_j}^{\Gamma} u^j + \sum_{j=1}^{n-1} h_{jj} \ \langle u,v\rangle$$
$$= \nabla^{\Gamma}.\ u^{\parallel} - H\langle u,v\rangle$$

Sothat $\nabla_n u^n = \Delta\,.\,u$

$$-\sum_{j=1}^{n-1} \nabla_j u^j = \Delta\,.\,u - \nabla^{\Gamma} u^{\parallel} + H\langle u,v\rangle \rightarrow (1.1.4)$$

$$= \Delta\,.\,u - \nabla^{\Gamma} u^{\parallel} + H\langle u,v\rangle \rightarrow (1.1.5)$$

Substitution (1.1.5) into (1.1.4)

$$\langle w.\nabla u, v\rangle = \langle w^{\parallel}, \nabla^{\Gamma}\langle u,v\rangle\rangle - \pi(u^{\parallel}, w^{\parallel}) + w^n(\nabla_n u^n)$$

$$= \langle w^{\parallel}, \nabla^{\Gamma}\langle u,v\rangle\rangle - \pi(u^{\parallel}, w^{\parallel}) + w^n(\Delta\,.\,u - u^{\parallel} + H\langle u,v\rangle)$$

$$\langle w.\nabla u, v\rangle = \langle w^{\parallel}, \nabla^{\Gamma}\langle u,v\rangle\rangle - \pi(u^{\parallel}, w^{\parallel}) + \langle w.v\rangle(\Delta\,.\,u - \nabla^{\Gamma} u^{\parallel} + H\langle u,v\rangle)$$

Which together with the identity:

$$\langle w.v\rangle\nabla^{\Gamma}.\ u^{\parallel} = \left(\nabla^{\Gamma}.\langle w,v\rangle\, u^{\parallel}\right) - \langle u^{\parallel}, \nabla^{\Gamma}\langle w,v\rangle\rangle$$

Hence the proof

1.2 Theorem

Let u be a vector field on Ω , $\omega = \nabla \times u$,and $d = \nabla.u$ .Then

(i) $\partial_v d\,\big|_{\Gamma} = \langle \Delta u, v\rangle + \nabla^{\Gamma} \times w^{\parallel}$

,where $\nabla^{\Gamma} \times w^{\parallel} = {\nabla^{\Gamma}}_1 w^2 - {\nabla^{\Gamma}}_2 w^1$ Is independent of the choice of a local moving frame ,which can be identified , with the exterior derivative of $w^{\parallel}$on Γ;

(ii) $\frac{1}{2}\partial_v \left(|u|^2\right)\big|_{\Gamma} = \langle u \times w, v\rangle + \langle u,v\rangle d - \pi(u^{\parallel}, u^{\parallel}) - H\langle u,v\rangle^2 + 2\langle u^{\parallel}\ \nabla^{\Gamma}.\langle u,v\rangle\rangle - \nabla^{\Gamma}.(\langle u,v\rangle u^{\parallel})$

Proof of (*i*) :

Given:

Let u be a vector field on Ω , $\omega = \nabla \times u$ and $d = \nabla.u$

To prove:

$$\partial_v d\,\big|_{\Gamma} = \langle \Delta u, v\rangle + \nabla^{\Gamma} \times w^{\parallel} \text{ ,where}$$
$$\nabla^{\Gamma} \times w^{\parallel} = {\nabla^{\Gamma}}_1 \omega^2 - {\nabla^{\Gamma}}_2 \omega^1$$

According to the vector identity:

$$\Delta u = -\nabla \times \omega + \nabla d,$$
$$\partial_v d\,\big|_{\Gamma} = \langle \Delta u, v\rangle + \langle \nabla \times w, v\rangle$$

It remains to verify that $\langle \nabla \times w, v\rangle$ coincides with $\nabla^{\Gamma} \times w^{\parallel}$ with depends only on the tangent part $w^{\parallel}\big|_{\Gamma}$.

Proof of (*ii*):

Given:

Let u be a vector field on Ω , $\omega = \nabla \times u$,and $d = \nabla.u$

To prove:

$$\frac{1}{2}\partial_v(|u|^2)\big|_{\Gamma} = \langle u \times w, v\rangle + \langle u,v\rangle d - \pi(u^{\parallel}, u^{\parallel}) - H\langle u,v\rangle^2 + 2\langle u^{\parallel}\ \nabla^{\Gamma}.\langle u,v\rangle\rangle - \nabla^{\Gamma}.(\langle u,v\rangle u^{\parallel})$$

This formula follows directly from the vector identity:

$\frac{1}{2}\Delta|u|^2 = u \times (\nabla \times u) + (u.\nabla)u,$

And using by theorem,

Let u and w be two vector fields on .

Then $\langle w.\nabla u, v\rangle = -\pi(u^{\parallel}, w^{\parallel}) - H\langle w,v\rangle\langle u,v\rangle + \langle w,v\rangle(\nabla.u)$

$+\langle w^{\parallel}, \nabla^{\Gamma}\langle u,v\rangle\rangle + \langle u^{\parallel}, \nabla^{\Gamma}\langle u,v\rangle\rangle - \nabla^{\Gamma}.(\langle w,v\rangle u^{\parallel})$

In particular ,if $u^{\perp} = w^{\perp} = 0$ on Γ ,

$$\langle w.\nabla u, v\rangle = -\pi(u,w)$$

Hence the proof

1.3 Theorem

If $u \in H^2(\Omega)$ is a vector field ,then

$$\int_{\Omega}\langle\Delta u,u\rangle dx = -\int_{\Omega}|\omega|^2 dx - \int_{\Omega}|\nabla.u|^2 dx + \int_{\Gamma}\langle u\times w, v\rangle ds + \int_{\Gamma}(\nabla.u)\langle u,v\rangle ds,$$

And

$$\int_{\Omega}|\nabla u|^2 dx = \int_{\Omega}|\omega|^2 dx + \int_{\Omega}|\nabla.u|^2 dx - \int_{\Gamma}(u^{\parallel}, u^{\parallel})ds - \int_{\Gamma} H|u^{\perp}|^2 ds + 2\int_{\Gamma}\langle u^{\parallel}, \nabla^{\Gamma}\langle u,v\rangle\rangle ds.$$

Proof :

Given:

If $u \in H^2(\Omega)$ is a vector field ,

To prove:

$$\int_{\Omega}\langle\Delta u,u\rangle dx = -\int_{\Omega}|\omega|^2 dx - \int_{\Omega}|\nabla.u|^2 dx + \int_{\Gamma}\langle u\times w, v\rangle ds + \int_{\Gamma}(\nabla.u)\langle u,v\rangle ds$$

$$\int_{\Omega}|\nabla u|^2 dx = \int_{\Omega}|\omega|^2 dx + \int_{\Omega}|\nabla.u|^2 dx - \int_{\Gamma}(u^{\parallel}, u^{\parallel})ds - \int_{\Gamma} H|u^{\perp}|^2 ds + 2\int_{\Gamma}\langle u^{\parallel}, \nabla^{\Gamma}\langle u,v\rangle\rangle ds.$$

This equation follows the vector identity

$$\Delta u = -\nabla\times\omega + \nabla(\nabla.u)$$

And integration by parts formula ,

$$\int_{\Omega}\langle\nabla\times u, w\rangle dx = \int_{\Omega}\langle u, \nabla\times w\rangle dx + \int_{\Omega}\langle u\times w, v\rangle ds$$

And the Bochner's identity

$$\langle\Delta u, u\rangle = \frac{1}{2}\Delta|u|^2 - |\nabla u|^2,$$

Together with integration by parts formula, we obtain

$$\int_{\Omega}\langle\Delta u,u\rangle dx = \int_{\Omega}|\nabla u|^2 dx + \frac{1}{2}\int_{\Gamma}\partial_v(|u|^2)\, ds$$

$$\int_{\Omega}\langle\Delta u,u\rangle dx = \int_{\Omega}|\nabla u|^2 dx + \int_{\Gamma}\langle u^{\parallel}\times w^{\parallel}, v\rangle ds + \int_{\Gamma}\langle u,v\rangle\nabla.u ds + 2\int_{\Gamma}\langle u^{\parallel}, \nabla^{\Gamma}\langle u,v\rangle\rangle ds - \int_{\Gamma}\pi(u^{\parallel}, u^{\parallel})ds - \int_{\Gamma}H\langle u,v\rangle^2 ds$$

Hence the proof

1.4 Theorem

There exists $C > 0$ depending only on Ω such that $\|\nabla^2 u\|_2^2 \le C(\|\Delta u\|_2^2 + \|\nabla u\|_2^2)$ for any vector field $u \in H^2$

Proof:

Given:

There exists $C > 0$ depending only on Ω

To prove :

$\|\nabla^2 u\|_2^2 \le C(\|\Delta u\|_2^2 + \|\nabla u\|_2^2)$ for any vector field $u \in H^2$

Applying the theorem

$\frac{1}{2}\partial_v \left(|u|^2\right)\big|_\Gamma = \langle u\times w, v\rangle + \langle u,v\rangle d - \pi(u^{\|}\, u^{\|}) - H\langle u,v\rangle^2 + 2\langle u^{\|}\ \nabla^\Gamma.\langle u,v\rangle\rangle - \nabla^\Gamma.(\langle u,v\rangle\, u^{\|})$

To gradient field ∇f forany scalar function f in Ω ,then

$$\frac{1}{2}\partial_v \left(|\nabla f|^2\right) = -\pi((\nabla f)^{\|}(\nabla f)^{\|}) - H|\partial_v f|^2 + \partial_v f\Delta f$$

$$+2\langle(\nabla f)^{\|}, \nabla^\Gamma(\partial_v f)\rangle - \nabla^\Gamma.(\partial_v f(\nabla f)^{\|}) \rightarrow 1.4.1$$

Write $u = (u^1, \dots\dots, u^n)$ under an orthonormal basis of $T\Omega$. then, according to the Bochner identity,

$$\|\nabla^2 \mathrm{u}\|_2^2 = \sum_{k=1}^{n} |\nabla^2 u^k|^2$$

$$= \sum_{k=1}^{n}\left(\frac{1}{2}\Delta|\nabla u^k|^2 - \langle\nabla\Delta u^k, \Delta u^k\rangle\right)$$

And ,the integration ,we find

$$\|\nabla^2 \mathrm{u}\|_2^2 = \sum_{k=1}^{n}\left(\frac{1}{2}\int_\Omega \Delta\, |\nabla u^k|^2 dx - \int_\Omega \langle\nabla\Delta u^k, \Delta u^k\rangle\right)$$

$$= \sum_{k=1}^{n}\left(\int_\Omega (\Delta u^k)^2 dx - \int_\Gamma \sum_{i,j=1}^{n-1} h_{ij}\,(\nabla_i u^k)(\nabla_j u^k) ds\right.$$

$$-\int_\Gamma \sum_{k=1}^{n} (|\partial_v u^k|^2) H ds$$

$$+ 2\int_\Gamma \sum_{k=1}^{n} \langle\nabla^\Gamma u^k, \nabla^\Gamma(\partial_v u^k)\rangle\, ds$$

$$= \int_\Gamma \sum_{i,j=1}^{n-1} h_{ij}\,(\nabla_i u^k)(\nabla_j u^k) ds$$

$$-\int_\Gamma \sum_{k=1}^{n} \langle\nabla u^k, v\rangle^2\, H ds$$

$$+2\int_\Gamma \sum_{i,j=1}^{n-1} \langle\nabla^\Gamma u^k, \nabla^\Gamma\langle\nabla u^k, v\rangle\rangle\, ds \ + \|\nabla^2 \mathrm{u}\|_2^2$$

Where the third equality follows from equation (1.4.1) applying to each u^k.

We handle the boundary integral:

$$\sum_{k=1}^{n}\int_\Gamma \langle\nabla^\Gamma u^k, \nabla^\Gamma(\partial_v u^k, v)\rangle ds$$

$$= \sum_{k=1}^{n-1}\int_\Gamma \langle\nabla^\Gamma u^k, \nabla^\Gamma(\partial_v u^k)\rangle ds$$

$$+ \int_\Gamma \langle\nabla^\Gamma u^n, \nabla^\Gamma(\partial_v u^n)\rangle ds$$

$$= \sum_{k=1}^{n-1}\int_\Gamma \langle\nabla^\Gamma u^k, \nabla^\Gamma(\partial_v u^k)\rangle ds$$

Where we have used the fact $u^n\big|_\Gamma = 0$ so that $\nabla^\Gamma u^n$=0.

For $1 \le k \le n-1$,we have $\nabla_n u^k = \nabla_k u^n = 0$ on Γ .

However,

$$\partial_v u^k = \nabla_n u^k - \sum_{j=1}^{n-1} u^j \Gamma^k{}_{nj}$$

$$= -\sum_{j=1}^{n-1} u^j \Gamma^k{}_{nj}$$

$$\sum_{j=1}^{n-1}\langle\nabla^\Gamma u^k, \nabla^\Gamma(\partial_v u^k)\rangle$$

$$= -\sum_{j=1}^{n-1}\langle\nabla^\Gamma u^k, \nabla^\Gamma \sum_{j=1}^{n-1} u^j \Gamma^k{}_{nj}\rangle$$

$$\le C(|u|^2 + |\nabla u|^2)$$

$$\sum_{j=1}^{n-1}\langle\nabla^\Gamma u^k, \nabla^\Gamma(\partial_v u^k)\rangle \le C(|u|^2 + |\nabla u|^2)$$

Hence the proof

2. Well – Posedness of the Stoke Equation and L^2-Estimate

In this, we discuss about some theorems on well-posedness of the stoke equation and L^2-estimate

2.1 INTRODUCTION

The initial –boundary value problem

$$\begin{cases} \partial_t w = \mu\Delta w - \nabla q\,, \quad \nabla.w = 0 \\ w^\perp\big|_\Gamma = 0\,, \quad (\nabla\times w)^{\|}\big|_\Gamma = a \\ w^\perp\big|_{t=0} = u_0 \end{cases}$$

And the scalar function q by solving the Neumann boundary problem

$$\Delta q = 0 \ , \quad \partial_v d \big|_\Gamma = -\mu \, \nabla^\Gamma \times a$$

Since $\int_\Omega \nabla^\Gamma \times a dS = 0$. There exist a unique solution q modulo a constant .

We renormalize q so that $\int_\Omega q dx = 0$

It suffices to the liner parabolic equation:

$$\begin{cases} \partial_t w = \mu\Delta w - \nabla q \,, \\ w^\perp \big|_\Gamma = 0 \,, \qquad (\nabla \times w)^\parallel \big|_\Gamma = a \\ w^\perp \big|_{t=0} = u_0 \end{cases}$$

Where q is given by $\Delta q = 0$, $\partial_v d \big|_\Gamma = -\mu \, \nabla^\Gamma \times a$,which is well-posed.

Taking the curl operation $\nabla \times$ in the equation in

$$\begin{cases} \partial_t w = \mu\Delta w - \nabla q \,, \quad \nabla . w = 0 \\ w^\perp \big|_\Gamma = 0 \,, \qquad (\nabla \times w)^\parallel \big|_\Gamma = a \\ w^\perp \big|_{t=0} = u_0 \end{cases}$$

With $g = \nabla \times w$ and $h = \nabla \times g = -\Delta w$,

$$\partial_t g = \mu \Delta g \ ; \ \partial_t h = \mu \Delta h \,,$$

Together with the boundary condition:

$$g^\parallel \big|_\Gamma = a \,, (\nabla \times g)^\perp \big|_\Gamma = \nabla^\Gamma \times a \,,$$

$$h^\perp \big|_\Gamma = \nabla^\Gamma \times a \,, (\nabla \times h)^\parallel \big|_\Gamma = -\frac{1}{\mu} \partial_t a$$

We now make the L^2- estimate for g and h .

$$\partial_t(|g|^2) = 2\mu\langle g, \Delta g\rangle \,, \ \partial_t(|h|^2) = 2\mu\langle h, \Delta h\rangle,$$

We integrate over Ω to obtain

$$\frac{d}{dt}\|g\|_2^2 = -2\mu \int_\Omega \langle g, \nabla \times (\nabla \times g)\rangle dx$$

$$= -2\mu \int_\Omega \langle \nabla \times g, \nabla \times g\rangle d \\ - 2\mu \int_\Gamma \langle (\nabla \times g)^\parallel \times a \,, v\rangle ds$$

$$\|g\|_2^2(t) + 2\mu \int_0^t \|\nabla \times h\|_2^2 \,(s) ds = \|g\|_2^2(0)$$

$$-2\mu \int_0^t \int_\Gamma \langle \left(\nabla \times g^\parallel\right) \times a, v\rangle \, dSds$$

For0 $\leq t \leq T$. → 2.1.1

$$\frac{d}{dt}\|h\|_2^2 = -2\mu \int_\Omega \langle h, \nabla \times (\nabla \times g)\rangle dx$$

Then we have,

$$= -2\mu \int_\Omega \langle \nabla \times h, \nabla \times h\rangle dx \\ - 2\mu \int_\Gamma \langle (\nabla \times h)^\parallel \times a \,, v\rangle ds$$

We have

$$\|h\|_2^2(t) + 2\mu \int_0^t \|\nabla \times h\|_2^2 \,(s) ds = \|h\|_2^2(0)$$

$$+2 \int_0^t \int_\Gamma \langle (\partial_t a) \times h, v\rangle \, dSds$$

For o $< t \leq T$. → 2.1.2

Hence the proof

2.2 Theorem

Let $w \in H^3(\Omega)$ be a solution of problem

$$\begin{cases} \partial_t w = \mu\Delta w - \nabla q \,, \quad \nabla . w = 0 \\ w^\perp \big|_\Gamma = 0 \,, \qquad \nabla \times w)^\parallel \big|_\Gamma = a \\ w^\perp \big|_{t=0} = u_0 \end{cases}$$

Then(i) $\partial_t a = 0$ and $\sqrt{\mu a} \in L^2(\Gamma_T)$, then there exists $C > 0$, independent of μ ,such that

$$\|w\|^2{}_{H^2(\Omega)}(t) + \mu\|w\|^2{}_{H^3(\Omega T)} \\ \leq C\|u_0\|^2{}_{H^2(\Omega)} + \mu\|a\|^2{}_{L^2(\Gamma T)}$$

(ii) $\partial_t a \neq 0$ and $(a, \ \partial_t a) \in(\Gamma_T)$,there exists

$$M = M(\mu, T) > 0$$

such that

$$\|w\|_{H^2(\Omega)}(t) + \|w\|^2{}_{H^3(\Omega T)}(t)$$

$$\leq M(\mu, T)\left(\|u_0\|^2{}_{H^2(\Omega)} + \|a, \ \partial_t a\|^2{}_{L^2(\Gamma T)}\right)$$

Proof of (i):

Given:

Let $w \in H^3(\Omega)$ be a solution of problem

$$\begin{cases} \partial_t w = \mu\Delta w - \nabla q \,, \quad \nabla . w = 0 \\ w^\perp \big|_\Gamma = 0 \,, \qquad (\nabla \times w)^\parallel \big|_\Gamma = a \\ w^\perp \big|_{t=0} = u_0 \end{cases}$$

To prove:

$$\|w\|^2{}_{H^2(\Omega)}(t) + \mu\|w\|^2{}_{H^3(\Omega T)} \\ \leq C\|u_0\|^2{}_{H^2(\Omega)} + \mu\|a\|^2{}_{L^2(\Gamma T)}$$

$\partial_t a = 0$ and $\sqrt{\mu a} \in L^2(\Gamma_T)$, for $\Gamma_T = \Gamma \times (0,T]$

We obtain from the equation

$$\|h\|_2^2(t) + 2\mu \int_0^t \|\nabla \times h\|_2^2(s)ds = \|h\|_2^2(0)$$

$$+2\int_0^t \int_\Gamma \langle(\partial_t a) \times h, v\rangle\, dSds$$

$$\|h\|_2^2(t) + 2\mu \int_0^t \|\nabla \times h\|_2^2(s)ds = \|h\|_2^2(0) \le \|w(0,.)\|_{H^2}$$

$$\|h\|_2^2(t) + 2\mu \int_0^t \|\nabla \times h\|_2^2(s)ds = \|u_0\|_{H^2}$$

$$\|h\|_2^2(t) + 2\mu \int_0^t \|\nabla \times h\|_2^2(s)ds = \|u_0\|_{H^2}$$

since $\nabla . g = \nabla . h = 0$,we conclude

$$\|\nabla g\|^2_{L^2(\Omega)}(t) + \mu\|\nabla^2 g\|^2_{L^2(\Omega T)} \le C \quad \text{for } t \in [o,T] \quad \to (2.2.1)$$

Where C > 0 is independent of μ.

Then we have

$$2\mu \left|\int_0^t \int_\Gamma \langle(\nabla \times g)^{\parallel} \times a, v\rangle\, dSds\right|$$

$$\le 2\mu\left(\varepsilon\int_0^t \int_\Gamma |\nabla g|^2\, dSds + C_\varepsilon \|a\|^2_{L^2(\Gamma T)}\right)$$

$$\le \mu\int_0^t \int_\Gamma |\nabla \times g|^2\, dxds + C\,\mu\|\nabla^2 g\|^2_{L^2(\Omega T)} + \mu\|a\|^2_{L^2(\Gamma T)}$$

$$\le \mu\int_0^t \int_\Omega |\nabla \times g|^2\, dxds + C\,\|u_0\|^2_{H^2(\Omega)} + \mu\|a\|^2_{L^2(\Gamma T)}$$

Substituting the equation in (2.1.1)

$$\|g\|_2^2(t) + 2\mu \int_0^t \|\nabla \times h\|_2^2(s)ds = \|g\|_2^2(0)$$

$$-2\mu \int_0^t \int_\Gamma \langle(\nabla \times g^{\parallel}) \times a, v\rangle\, dSds$$

For $0 \le t \le T$.

$$\|g\|^2_{L^2(\Omega)} + \mu\|\nabla g\|^2_{L^2(\Omega T)} \le C\,\|u_0\|^2_{H^2(\Omega)}$$

$$+\mu\|a\|^2_{L^2(\Gamma T)} \to (2.2.2)$$

Comparing this into equation (2.2.1) – (2.2.2)

With fact that $\nabla . w = \nabla . g = \nabla . h = 0$, we have

$$\|w\|^2_{H^2(\Omega)}(t) + \mu\|w\|^2_{H^3(\Omega T)} \le C\|u_0\|^2_{H^2(\Omega)} + \mu\|a\|^2_{L^2(\Gamma T)}$$

Where C > 0 is independent

Hence the proof

Proof of (ii):

Given:

Let $w \in H^3(\Omega)$ be a solution of problem

$$\begin{cases} \partial_t w = \mu\Delta w - \nabla q\,, \quad \nabla . w = 0' \\ w^{\perp}\big|_\Gamma = 0\,, \quad \nabla \times w)^{\parallel}\big|_\Gamma = a \\ w^{\perp}\big|_{t=0} = u_0 \end{cases}$$

To prove:

$$\|w\|_{H^2(\Omega)}(t) + \|w\|^2_{H^3(\Omega T)}(t) \le M(\mu,T)\left(\|u_0\|^2_{H^2(\Omega)} + \|a,\, \partial_t a\|^2_{L^2(\Gamma T)}\right)$$

$\partial_t a \neq 0$ and $(a,\, \partial_t a) \in (\Gamma_T)$,

we have the

$$2\left|\int_0^t \int_\Gamma \langle\, \partial_t a \times h, v\rangle\, dSds\right|$$

$$\le \varepsilon\int_0^t \int_\Gamma |h|^2\, dSds + C_\varepsilon\|\,\partial_t a\|^2_{L^2(\Gamma T)}$$

$$\le \mu\int_0^t \int_\Gamma |\nabla \times h|^2\, dxds + \frac{C}{\mu}\int_0^t \int_\Gamma |h|^2\, dSds + C\,\|\,\partial_t a\|^2_{L^2(\Gamma T)}$$

Substituting the equation (2.1.2)

$$\|h\|_2^2(t) + 2\mu \int_0^t \|\nabla \times h\|_2^2(s) = \|h\|_2^2(0)$$

$$+2\int_0^t \int_\Gamma \langle(\partial_t a) \times h, v\rangle\, dSds \text{ For} 0 \le t \le T.$$

$$\|h\|_2^2(t) + 2\mu \int_0^t \|\nabla \times h\|_2^2(s)ds \le C\|u_0\|^2_{H^2(\Omega)} + C\|\partial_t a\|^2_{L^2(\Gamma T)} + \frac{C_0}{\mu}\int_0^t \|h\|_2^2(s)ds$$

Then the Gronwall inequality yields

$$\|h\|_2^2(t) \leq M(\mu,T)\left(\|u_0\|^2{}_{H^2(\Omega)} + \|a,\ \partial_t a\|^2{}_{L^2(\Gamma T)}\right) \to \ 2.2.1$$

$$\|g\|_2^2(t) + 2\mu\int_0^t \|\nabla \times h\|_2^2\,(s)ds = \|g\|_2^2(0)$$

$$= \|g\|_2^2(0) - 2\mu\int_0^t \int_\Gamma \langle(\nabla \times g^{\|}) \times a, v\rangle\, dSds$$

For $0 < t \leq T.$ → 2.2.2

Combining the equation (2.2.1) and (2.2.2)

$$\|w\|_{\ H^2(\Omega)}(t) + \|w\|^2{}_{H^3(\Omega T)}$$

$$\leq M(\mu,T)\left(\|u_0\|^2{}_{H^2(\Omega)} + \|a,\ \partial_t a\|^2{}_{L^2(\Gamma T)}\right)$$

Where $M(\mu,T) > 0$ depends only on μ and T

Hence the proof

CONCLUSION

Throughout this work, we discussed some definition and theorems on Navier-stoke equation with the kinematic and vorticity boundary condition on non-flat boundary .and then we discussed a well-posedness of the stoke equation and L^2 **-** estimate. Finally,we establish that ,when the viscosity coefficient tends zero, the strong solutions of the initial-boundary value problem in $\mathbb{R}^n$ with non homogeneous vorticity boundary condition converge in L^2**.**

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