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The Three dimensional Navier stokes equations on Global attractor

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Abstract

We introduce the three dimensional Navier Stokes Equations might not always have regular solutions we introduce an abstract framework for studying the asymptotic behavior of multi-valued dissipative evolutionary systems with respect to two topologies weak and strong. Each such system possesses a global attractor in the weak topology, but not necessarily in the strong. These models always possess weak global attractors, but on some of them every solution blow up in finite time. In the context of the dyadic Navier-Stokes equations with hyper-dissipation we prove finite time blow-up in the case when the dissipation degree is sufficiently small.

Keywords Navier-Stokes equations,Global atteractor, Blow- up in finite time, Dyadic Navier-Stokes equations, Hyper-dissipation.

INTRODUCTION

A remarkable feature of many dissipative partial differential equations is the existence of a global attractor to which all the solutions converge as time goes to infinity. The global attractor $\mathcal{A}$ is the minimal closed set in a phase space H(i.e., the functional space) that uniformly attracts the trajectories starting from any a priori given bounded set in H. It is possible that a dissipative PDE does not have a strong global attractor. For instance, the 2D Navier-Stokes equations (NSE) on a bounded domain $\Omega \subset \mathbb{R}$, when supplemented with appropriate boundary conditions, possess a strong global attractor in H(a certain subspace of $L^2(\Omega)^3$), but it is not yet known this holds for the 3D NSE.

Nevertheless, even for the 3D NSE one can prove that there exists a weak global attractor. When the strong global attractor is strongly compact in H(e.g., in the 2D NSE), then it is also the weak global attractor. But, in any case, the weak global attractor is an appropriate generalization of the strong global attractor since it captures the long-time behavior of the solutions. In particular, the support of any time- average measure of the 3D NSE is included in the weak global attractor(see[3]). One should note that Sell introduced a related notion of a trajectory attractor $\mathfrak{A}$ in the space of all trajectories. The weak global attractor coincides with the set of values of all trajectories in $\mathfrak{A}$ at any fixed time t.

The aim of this study is to present a general abstract framework which is applicable to the 3D NSE even in the case where they do not possess a strong global attractor. This framework may be also useful in the study of the other PDEs for which the existences of the strong global attractor is in limbo. This aim forces us to consider multi-valued evolutionary systems. The main difference between our evolutionary system ε and Ball's generalized semi flow is that we do not include the hypotheses of concatenation and upper semi continuity with respect to the initial data. This allows us to consider an evolutionary system whose trajectories are all Leary-Hopf weak solutions of the 3D NSE.

Our definition of the evolutionary system ε already exploits the effect of dissipativity, namely the existence of an absorbing ball. In fact, the space X in which the trajectories of ε live is, in applications, precisely such an absorbing ball.

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We show that every evolutionary system always possesses a weak global attractor; moreover, if the strong global attractor exists and is weakly closed, then it has to coincide with the weak global attractor. Note that some classical definitions require a global attractor to be an invariant set. We will see tat under a condition, which is, satisfied by the Leary-Hopf weak solutions of the 3D NSE, the weak global attractor is also the maximal bounded invariant set.

In this paper we show that even without the assumptions of concatenation and upper semi continuity with respect to the initial data, the asymptotic compactness implies that the weak global attractor is the minimal compact attracting set in the strong metric, i.e., the weak global attractor is in fact the strong compact global attractor. Applied to the 3D NSE, this result implies the existences of a strong compact global attractor in the case when the solutions on the weak global attractor are continuous in $L^2(\Omega)^3$.

We provide an example of a dissipative evolutionary system for which all solutions on the weak global attractor blow- up in finite time. We introduce a two- parameter family of simple infinite –dimensional systems of differential equations. These systems, called tridiagonal models for the Navier-Stokes equation(TNS models), display basic features of the NSE. In particular they are examples of dissipative systems that possess a weak global attractor , but the existence of a strong global attractor is not known.

TNS models have similar form and some similar properties to shell models, specifically dyadic models studied in [4, 6]. Moreover, a similar analysis of the dyadic models also results in a finite time blow-up. Then we obtain the following system of differential equations:

$$\frac{d}{dt}(u) + \nu Au + B(u,u) = g \qquad \rightarrow(1.1)$$

where u = $(u_1, u_2, \ldots\ldots)$,

$(Au)_n = n^\alpha u_n$, and

$$\big(B(u,v)\big)_n = -n^\beta u_{n-1}v_{n-1} + (n+1)^\beta u_n v_{n+1},$$

n = 1, 2, ..., with $u_0 = 0$.

Here $\alpha\ and \beta$ are two positive parameters. Note that the orthogonality property in l^2 holds for B, which implies the existence of an absorbing ball and a weak global attractor. Moreover, when $\alpha = 2/3$, which corresponds to the speed with which the eigenvalues of the stokes operator grow in three-dimensional space, and $\beta = 11/6$, we have the following sharp estimate:

$$|(B(u,u), Au)| \leq |Au|^{3/2}\left|A^{1/2}u\right|^{3/2},$$

Where $|v|^2 = \sum v_n^2$.

Finally, the dyadic model presented here is an infinite system of nonlinearly coupled ODEs. Each ODE illustrates time evolution of a wavelet coefficient which describes behavior of the velocity that is localized to a certain frequency range. Therefore, the dyadic model could be understood in a general context of Littlewood- Paley theory, which

allows localization of a function into frequency range. Recently Littlewood- Paley theory was extensively used in studying the Navier-Stokes equation. Each ODEs in our model reflects behavior of the velocity localized in space on a dyadic cube. Here we used a standard harmonic analysis approach of decomposing $\mathbb{R}^3$ into dyadic cubes. Using such a decomposition we study flows on the dyadic tree.

We first used dyadic models in our work on partial regularity for the Navier-Stokes equations with hyper-dissipation[10] where they were a helpful test case for our ideas. On the other hand, by following harmonic analysis techniques one can obtain existence results for the dyadic models that correspond to the known existence results for the actual Euler and Navier-Stokes equations. We present a proof of finite time blow-up for the dyadic Navier-Stokes equations with hyper-dissipation.

2. Evolutionary system and Global attractors

Let (X, $d_{s(.,.)}$) be a metric space with a metric d_s, which will be referred to as a strong metric.

Let $d_{w(.,.)}$ be another metric on X satisfying the following conditions:

1. X is d_w compact
2. If $d_s(u_n, v_n) \longrightarrow 0$ as $n \longrightarrow \infty$ for some u_n, v_n $\in X$,the $d_w(u_n, v_n) \longrightarrow 0$ as $n \longrightarrow \infty$,that is the identity map(X, d_s) $\longrightarrow$ (X, d_{ws}) is uniformly continuous.

2.1 Definition

A map $\mathcal{E}$ that associates to each I∈T a subset $\mathcal{E}$ (I) ⊂ F will be called an evolutionary system if the following conditions are satisfied:

1. $\mathcal{E}([0, \infty)) \neq \emptyset$
2. $\mathcal{E}(I+s) = \{u(.):u(.-s) \in \mathcal{E}(I)\}\ \forall\ s \in \mathcal{R}$
3. $\{u(.)|_{I_2}: u(.) \in \mathcal{E}(I_1)\} \subset \mathcal{E}(I_2)\ \forall$ pairs $I_1, I_2 \in T$ such that $I_2 \subset I_1$.
4. $\mathcal{E}(-\infty, \infty) = \{u(.): u(.)|_{[T,\infty)} \in \mathcal{E}([T, \infty))\ \forall\ T \in \mathcal{R}\}$.

2.2 Definition

A set A ⊂X is a $d_{(s,w)}$-attracting set if it uniformly attracts X in $d_{(s,w)}$-metric. ie) For any ∈>0 there

2.3 Definition

$A_{(s,w)}$ ⊂X is a $d_{(s,w)}$-global attractor if $A_{(s,w)}$ is a minimal $d_{(s,w)}$-closed $d_{(s,w)}$-attracting set.ie) $A_{(s,w)}$ is $d_{(s,w)}$-closed $d_{(s,w)}$-attracting and every subset A⊂ $\mathcal{A}_{(s,w)}$ that is also $d_{(s,w)}$-closed and $d_{(s,w)}$-attracting satisfies A= $\mathcal{A}_{(s,w)}$.

2.4 Lemma

If $\mathcal{A}_{(s,w)}$ exists and A is a $d_{(s,w)}$-closed $d_{(s,w)}$-attracting set, then $\mathcal{A}_{(s,w)}$ ⊂A.

Proof:

Take any point a ∈ $\mathcal{A}_{(s,w)}$

Let∈>0. If there exists t ∈ >0 such that R(t) X ∩ $B_{(A,\ \in)}$=∅,∀ t≥t∈ then $\mathcal{A}_{(s,w)}|_{B(a,\ \in/2)}$ is a $d_{(s,w)}$-closed $d_{(s,w)}$-attracting set.

Contradicting the minimality of $\mathcal{A}_{(s,w)}$. So, there exists a sequence $t_n \longrightarrow \infty$ as $n \longrightarrow \infty$ such that

$R(t_n)$ X ∩ $B_{(A,\ \in)}$≠ ∅,∀n.

Since A is $d_{(s,w)}$-attracting , then our definition,

$R(t_n)X \subset B_{(s,w)}(A, \epsilon)$

for n large enough. It follows that,

A ∩$B_{(A,\ \in)}$(a,2 ∈) ≠ ∅,

Since A is $d_{(s,w)}$-closed. We have a∈ A. Thus, $\mathcal{A}_{(s,w)}$ ⊂A.

2.4.1 Theorem

If $\mathcal{A}_s$ exists, then $\mathcal{A}_w$ exists and $\mathcal{A}_w = \bar{\mathcal{A}}_s^{\,w}$

Proof:

If there exists a d_w- exists d_w- attracting set A ⊂ $\bar{\mathcal{A}}_s^{\,w}$ and A≠ $\bar{\mathcal{A}}_s^{\,w}$ then there exists u_0 ∈ $\mathcal{A}_s$, such that $d = d_w(u_0, A) > 0$.

By the definition of an attracting set,there exists a time t_0>0, such that

$$R(t)x \subset B_w(A, d/2),\ \forall\ t \geq t_0 \qquad \rightarrow(2.1)$$

We known that $d_w(u_0, B_w(A, d/2)) \geq d/2$.

Therefore , by the definition of d_w, there exists δ>0 such that

$$d_s(u_0, B_w(A, d/2)) > \delta,$$

Hence $B_s(u_0, \delta) \cap B_w(A, d/2) = \emptyset$

Now from(2.1) follows that

$$B_s(u_0, \delta) \cap R(t)X = \emptyset, \forall\ t \geq t_0$$

Consequently, $\mathcal{A}_s \setminus B_s(u_0, \delta/2)$ is a d_s-closed d_s-attracting set strictly included in $\mathcal{A}_s$.Which is contradiction. Hence $\bar{\mathcal{A}}_s^{\,w}$ is the weak global attractor.

2.5 Definition

The $\omega_{(s,w)}$-limit of a set K⊂X is

$$\omega_{(s,w)}(K) = \bigcap_{T \geq 0} \overline{\bigcup_{t \geq T} R(t)K}^{(s,w)},$$

where the closure is taken in $d_{(s,w)}$-metric.

2.6 Lemma

Let A be a $d_{(s,w)}$-closed $d_{(s,w)}$-attracting set. Then $\omega_{(s,w)}(X) \subset$ A.

Proof:

Suppose that there exist a ∈ $\omega_{(s,w)}(X) \setminus A$. Since A is $d_{(s,w)}$-closed, there exist $\epsilon > 0$,such that

$$A \cap B_{(s,w)}(a, \epsilon) = \emptyset.$$

By the definition of the $\omega_{(s,w)}$-limit, there exist a sequence $t_n \to \infty$ as $n \to \infty$ and a sequence $x_n \in R(t_n)X$, such that $d_{(s,w)}(X_n, a) \to 0$ as $n \to \infty$.Hence, there exist N > 0, such that

$x_n \notin B_{(s,w)}(A, \epsilon/2)\ \forall\, n \geq N$.

This means that A is not $d_{(s,w)}$-attracting, a contradiction.

2.7 Lemma

If the map R(t) is uniformly $d_{(s,w)}$- compact then $\omega_{(s,w)}(X)$ is a non empty $d_{(s,w)}$- compact $d_{(s,w)}$-attracting set.

Proof:

By the definition of R(t) is uniformly $d_{(s,w)}$-compact and the fact that $\mathcal{E}([0,\infty)) \neq \emptyset$, there exist t_0, such that

$$W(T) = \bigcup_{t \geq T} R(T)X^{(s,w)}$$

is a nonempty $d_{(s,w)}$- compact set $\forall\ T \geq t_0$.In addition, $W(s) \subset W(t)\ \forall\ s \geq t \geq 0$.Thus,

$$\omega_{(s,w)}(X) = \bigcap_{T \geq t_0} W(T)$$

is a nonempty $d_{(s,w)}$- compact set.

Now to prove: $\omega_{(s,w)}(X)$ uniformly $d_{(s,w)}$-attracts X. Assume $\omega_{(s,w)}(X)$ does not uniformly $d_{(s,w)}$-attracts X. Then there exist $\epsilon > 0$,such that

$$V(t) = W(t) \cap (X \setminus B_{(s,w)}(\omega_{(s,w)}(X), \epsilon)) \neq \emptyset,\ \forall\ t \geq 0$$

Since V(t) is $d_{(s,w)}$- compact for $t \geq t_0$ and $V(s) \subset V(t)$ for $s \geq t0$,we have that there exist

$$x \in \bigcap_{t \geq t_0} V(t)$$

Hence, $x \in \omega_{(s,w)}(X)$.However, this implies that $x \notin V(t),\ t \geq 0$,a contradiction.

2.8. Definition

The set $A \subset X$ is invariant, if

$\{u(t): u \in \mathcal{E}(-\infty, \infty), u(0) \in A\} = A,\ \forall\, t \geq 0$.

2.9. Theorem

If $\mathcal{E}([0, \infty))$ is compact in$C([0, \infty); X_w)$.Then

a) $\mathcal{A}_w = I = \{u_0 : u_0 = u(0)$ for some $u \in \mathcal{E}(-\infty, \infty)\}$
b) $\mathcal{A}_w$ is the maximal invariant set.

Proof:

Since obviously I is the maximal invariant set. Then we have, $I \subset \mathcal{A}_w$.

It remains to prove $\mathcal{A}_w \subset I$. Let $a \in \mathcal{A}_w$. since $\mathcal{A}_w = \omega_{(s,w)}(X)$,there exist $t_n \to \infty$ as $n \to \infty$ and $a_n \in R(t_n)X$, such that $a_n \to a$ *weakly* as $n \to \infty$.

Using $\mathcal{E}(I + s) = \{u(.): u(.- s) \in \mathcal{E}(I)\}\ \forall\ s \in \mathcal{R}$, there exist $u_n \in \mathcal{E}([-t_n, \infty))$ such that $u_n(0) = a_n$. Also using the same property of $\mathcal{E}$ imply that $\mathcal{E}([-t_n, \infty))$ is compact in $C([-t_n, \infty): X_W)$ and

$\{u|[-t_1, \infty): u \in \mathcal{E}([-t_n, \infty))\} \subset \mathcal{E}([-t_1, \infty))$

for every n. Assume that $u_n|[-t_1, \infty) \to u^1 \in \mathcal{E}([-t_n, \infty))$ in $C([-t_1, \infty): X_W)$ as $n \to \infty$. By a standard diagonalization process we obtain that there exist u ∈ F((-∞,∞)) and a subsequence of u_n and denoted by u_n, such that $u_n|[-T, \infty) \to u_{[-T, \infty)}$ in $C([-T, \infty): X_W)\ \forall\ T > 0$.Thus, by the compactness we have that $u|[-T, \infty) \in \mathcal{E}([-T, \infty))\ \forall\ T > 0$.

Hence, $u \in \mathcal{E}([-\infty, \infty))$. Finally since u(0) = a, therefore we get , $a \in I$. Hence, $\mathcal{A}_w \subset I$.

2.10. Definition

The map R(t) is asymptotically $d_{(s,w)}$- compact if for any $t_n \to \infty$ and any $x_n \in R(t_n)X$, the sequence $\{x_n\}$ is relatively $d_{(s,w)}$- compact.

2.11. Theorem

If the map R(t)) is asymptotically d_s- compact, then $\mathcal{A}_w$ is d_s- compact strong global attractor.

Proof:

First, note that $\omega_s(X) \subset \omega_w(X) = \mathcal{A}_w$.

On the other hand, let $a \in \mathcal{A}_w = \omega_w(X)$. By the definition of ω_w- limit, there exist $t_n \to \infty$ as $n \to \infty$ and $x_n \in R(t_n)X$, such that

$$d_W(x_n, a) \to 0 \text{ as } n \to \infty.$$

This convergence is in fact strong .Therefore, $a \in \omega_s(X)$.

Hence, $\omega_s(X) = \mathcal{A}_w$

Now to show that: $\omega_s(X)$ is a d_s- attracting set. Assume that $\omega_s(X)$ is not d_s- attracting set. Then there exist $\epsilon > 0$, $x_n \in X$ and $t_n \to \infty$ as $n \to \infty$,such that

$x_n \in R(t_n)X \setminus B_s(\omega_s(X), \epsilon),\ \forall\ n \in N$.

Since R(t)) is asymptotically d_s- compact, then $\{x_n\}$ is relatively $d_{(s,w)}$- compact.We may assume that,

$x_n \to x \in X$ strongly as $n \to \infty$.

Therefore, we have that $x \in \omega_s(X)$, a contradiction.

Hence, $\omega_s(X)$ is a d_s- attracting set.

Since, $\omega_s(X)$ is the minimal d_s- closed d_s- attracting set (since by lemma(2.6)). Therefore, $\omega_s(X)$ is strong global attractor $\mathcal{A}_s$.

Finally to prove: $\omega_s(X)$ is strongly compact. Let any sequence $a_n \in \omega_s(X)$.By the definition of ω_s – limit, there exist $t_n \to \infty$ and $x_n \in R(t_n)X$, such that

$d_s(x_n, a_n)) \to 0$ as $n \to \infty$.

Since, $\{x_n\}$ is relatively d_s- compact due to the asymptotic compactness of R(t).Hence, $\{a_n\}$ is relatively d_s- compact.

3. 3Dimensional Navier stokes equations

The space periodic 3D Navier - Stokes equations

$$\frac{d}{dt}u - v\Delta u + (u.\nabla)u + \nabla p = f,$$

$$\nabla.u = 0$$

u, p, f are periodic with period L in each space variable

u, f are in $L^2_{loc}(R^3)^3$

$u|_{t=0} = u_0$ →(3.1)

where u the velocity, p the pressure ,are unknowns, f is a given driving force, and v > 0 is the kinematic viscosity co-efficient of the fluid. By a Galilean change of variables, we can assume that the space average of u is zero,

ie) $\int_{\Omega} u(x,t)dx = 0, \forall t,$

where $\Omega = [0, L]^3$ is a periodic box.

3.1 Functional setting

Denote (.,.)and |.| the $L^2(\Omega)^3$- inner product and the corresponding $L^2(\Omega)^3$- norm. Let γ be the space of all R^3 trigonometric polynomials of period Lin each variable satisfying $\nabla.u = 0$ and $\int_{\Omega} u(x)dx = 0$

Let H and V be the closures of γ in $L^2(\Omega)^3$ and $H^1(\Omega)^3$, respectively. Also, define the distance $d_{(s,w)}$ by

$d_s(u,v) = |u-v|$,

$$d_w(u,v) = \sum_{k\in z^3} \frac{1}{2^{|k|}} \frac{|u_k - v_k|}{1+|u_k - v_k|}, u, v \in H$$

where u_k and v_k are Fourier co-efficient of u and v, respectively.

Denote by $P_\sigma : L^2(\Omega)^3 \to H$ the L^2-orthogonal projection, and be $A = -P_\sigma \Delta = \Delta$ the stokes operator with the domain $D(A) = (H^2(\Omega))^3 \cap V$.

Denotes $\|u\| = |A^{1/2}u|$

$$= \left(\int_\Omega \sum_{i,j=1}^{3} \left| \frac{\partial u_i}{\partial x_j} \right|^2 dx \right)^{1/2}$$

Equation (3.1) now can be condensed in the functional differential equation

$$\frac{d}{dt} u + vAu + B(u,u) = g \text{ in} V^1 \qquad \to (3.2)$$

where V^1 be the set of all distribution of the form v = Δu with $u \in v$, and u is a V-valued function of time and $g = P_\sigma f$. Assume that g is time independent and $g \in H$.

3.1.1 Theorem (Leray, Hopf)

For every $u_0 \in H$, there exist a weak solution u(t) of (2.2) on [T, ∞) with $u(T) = u_0$ satisfying the following energy inequality:

$$|u(t)|^2 + 2v \int_{t_0}^{t} \|u(s)\|^2 ds \le |u(t_0)|^2 + 2 \int_{t_0}^{t} (g(s), u(s)) ds \qquad \to (3.3)$$

$\forall t \ge t_0$, t_0 almost everywhere in [T, ∞).

See [5] for a hypothesis under which the energy equality holds.

3.1.2 Definition

A Leary- Hopf solution of (2.2) on the interval [T, ∞) is a weak solution on [T, ∞) satisfying the energy inequality (2.5) for all $T \le t_0 \le t$, t_0 almost everywhere in [T, ∞).The set Ex of measure 0 of points t_0 for which the energy inequality does not hold will be called the exceptional se**t.** The solution u(t) will be called regular on an interval $(\alpha, \beta) \subset [T, \infty)$ if $u(t) \in v$ and $\|u(t)\|$ is continuous on (α, β).

3.1.3 Theorem

Let u(t) be a Leary-Hopf solution of (2.2) on [T, ∞).Then there are almost countably many distinct open intervals I_j, such that

$$[T, \infty) = \overline{\bigcup_j I_j}$$

u(t) is regular on each I_j, and the measure of $[T, \infty) \setminus \bigcup_j I_j$ is zero.

3.2 The weak global attractor for the 3D Navier Stokes equation

A ball $B \subset H$ is called an absorbing ball if for any bounded set $K \subset H$, there exist t_0 such that

$u(t) \in B$, $\forall t \ge t_0$

for all Leary-Hopf solution u(t) on (3.2) on [0, ∞) with $u(0) \in K$.

Consider an evolutionary system for which a family of all trajectories consists of all Leary-Hopf solutions of the 3D Navier -Stokes equations in X.

Define $\mathcal{E}([T, \infty)) = \{u(.): u(.)$ is a Leary-Hopf solution on [T, ∞) and $u(t) \in X, \forall t \in [T, \infty)\}, T \in \mathcal{R}$

$\mathcal{E}((\infty, \infty)) = \{u(.): u(.)$ is a Leary-Hopf solution on (-∞, ∞) and $u(t) \in X, \forall t \in (-\infty, \infty)\}$.

3.2.1 Lemma

If u(t), a Leary-Hopf solution of the 3D Navier-Stokes equation, satisfies $\limsup_{t\to\infty} \|u(t)\| < \infty$ then u(t) converges strongly in H to the weak global attractor $\mathcal{A}_w$.

Proof:

Suppose that u(t) does not converges strongly in H to $\mathcal{A}_w$.Then there exist M >0 and a sequence $t_n \to \infty$ as $n \to \infty$, such that

$$d_s(u(t_n), \mathcal{A}_w) > M, n \in N \qquad \to (3.4)$$

we know that there exist a time T > 0, such that $\{u(t): t \ge T\}$ is relatively compact in H. Therefore, passing to a subsequence, we may assume that $u(t_n)$ converges strongly in H to some $a \in H$.Therefore, $a \in \mathcal{A}_w$. Which contradicts (3.4).

3.3 Strong convergence of Leary-Hopf solution

3.3.1 Theorem

Let u(t) be the Leary-Hopf solution on [T, ∞). Let Ex be the exceptional set for this solution. Then for any time $t_0 > T$, there exist A_- and A_+, such that

a) For every sequence $\{t_n\} \subset [T, \infty) \setminus Ex$, such that $t_n \to t_0$, $t_n < t_0$ it follows that $|u(t_n)| \to A_-$ as $n \to \infty$.
b) For every sequence $\{t_n\} \subset [T, \infty) \setminus Ex$, such that $t_n \to t_0$, $t_n > t_0$ it follows that $|u(t_n)| \to A_+$ as $n \to \infty$.

Where, $A_- = \lim_{t\to t_0^-} |u(t)|$, $A_+ = \lim_{t\to t_0^+} |u(t)|$.

Proof:

For $\{t_n\}$as in (a), The energy inequality (3.3) on $[t_n, t_{n+k}]$ is

$$|u(t_{n+k})|^2 + 2v \int_{t_n}^{t_{n+k}} \|u(s)\|^2 ds \le |u(t_n)|^2 + 2 \int_{t_n}^{t_{n+k}} (g, u(s)) ds,$$

Provided $t_{n+k} \ge t_n$. Taking the upper limit as $k \to \infty$, we get

$$\limsup_{n\to\infty} |u(t_n)|^2 + 2v \int_{t_n}^{t_0} \|u(s)\|^2 ds \le |u(t_n)|^2 + 2 \int_{t_n}^{t_0} (g, u(s)) ds$$

Taking the lower limit as $n \to \infty$, we obtain

$$\limsup_{n\to\infty}|u(t_n)|^2 \le \liminf_{n\to\infty}|u(t_n)|^2$$

ie) $\lim_{n\to\infty}|u(t_n)|$ exists. Since the limit exists for any sequence t_n, it does not depend on the choice of sequence.

3.4 The strong global attractor for the 3D Navier-Stokes equation

3.4.2 Lemma

If all solutions on the weak global attractor are strongly continuous in H. i.e., if $\varepsilon((-\infty,\infty)) \subset C((-\infty,\infty);\mathrm{H})$, then R(t) is asymptotically compact.

Proof:

Take any $\{t_n\}$, such that $t_n \to \infty$ as $n \to \infty$, and $x_n \in R(t_n)X$. Then there exist a sequence of solutions $v_n \in \varepsilon([0,\infty))$, such that $v_n(t_n) = x_n$. To show that $\{x_n\}$ has a convergent subsequence. Without loss of generality,there exist T > 0, such that $t_n \ge 2T$ for all n.

Consider a sequence $u_n(t) = v_n(t + t_n - T)$, where t ≥ 0.Take $\varepsilon([0,\infty))$ is compact in $C([0,\infty); H_w)$. Hence, passing to a subsequence and dropping a sub index, we can assume that u_n converges to some $u \in \varepsilon([0,\infty))$ in $C([0,\infty); H_w)$ as $n\to\infty$.

By the definition of the weak global attractor, $u(t) \in A_w$ for all $t \in [0,\infty)$. Applying $(T_1,T_2) = (0, T)$, we obtain that there exists a complete trajectory $v \in \varepsilon((-\infty,\infty))$, such that u(t) = v(t) on $[T/2, \infty)$.

Therefore, $u \in C([T/2, \infty); H)$.

Hence, $u_n(t) \to u(t)$ strongly in H, i.e) $x_n \to u(t)$ strongly in H.

4. Tridiagonal models for the Navier-Stokes Equations

4.1 A Priori estimates and the existence of strong solutions

4.1.1 Theorem

Let u(t) be a solution of (1) with $u_n(0) \ge 0$.Then $u_n(t) \ge 0$ for all t > 0, and u(t) satisfies the energy inequality

$$|u(t)|^2 + 2v\int_{t_0}^{t}\|u(\tau)\|^2 d\tau \le |u(t_0)|^2 + 2\int_{t_0}^{t}(g,u(\tau))d\tau$$

for all $0 \le t_0 \le t$.

Proof:

A general solution for $u_n(t)$ can be written as

$$u_n(t)=u_n(0)\exp\left(-\int_0^t\left[vn^\alpha + (n+1)^\beta u_{n+1}(\tau)\right]d\tau\right) + \int_0^t\left(g_n + n^\beta u_{n-1}^2(s)\right)\exp\left(-\int_s^t\left[vn^\alpha + (n+1)^\beta u_{n+1}(\tau)\right]d\tau\right)ds$$

Since $u_n(0) \ge 0$ for all n, then $u_n(t) \ge 0$ for all n, t > 0.

Now, multiplying by u_n, we have

$$\frac{d}{dt}u_n(t)^2 - \frac{d}{dt}u_n(0)^2 + 2vn^\alpha u_n^2 + 2(n+1)^\beta u_n^2 u_{n+1} = 2g_n u_n$$

Taking summation 1 to N, we get

$$\sum_{n=1}^{N}\frac{d}{dt}u_n(t)^2 - \sum_{n=1}^{N}\frac{d}{dt}u_n(0)^2 + 2v\sum_{n=1}^{N}n^\alpha u_n^2 + 2(N+1)^\beta u_N^2 u_{N+1} = 2\sum_{n=1}^{N}g_n u_n$$

And integrating between t_0 and t, we get

$$\sum_{n=1}^{N}u_n(t)^2 - \sum_{n=1}^{N}u_n(0)^2 + 2v\int_{t_0}^{t}\sum_{n=1}^{N}n^\alpha u_n^2 d\tau = -2\int_{t_0}^{t}(N+1)^\beta u_N^2 u_{N+1}d\tau + 2\int_{t_0}^{t}\sum_{n=1}^{N}g_n u_n d\tau \le 2\int_{T_0}^{T}\sum_{n=1}^{N}g_n u_n$$

Taking limit as $N \to \infty$ we get the result.

4.2 Blow-up in Finite time

4.2.1 Lemma

Let u(t) be a solution to (1) on $[0,\infty)$ with $u_n(0) \ge 0$ for all n. Assume that $\|u(t)\|_{2(\beta+\gamma-1)/3} \in L^3_{loc}([0,\infty); R)$.Then

$$\left.\begin{aligned}&\int_{t_0}^{t}\sum_{n=1}^{\infty}n^{\beta+\gamma-1}u_n^2 u_{n+1}d\tau < \infty,\\ &\int_{t_0}^{t}\sum_{n=1}^{\infty}n^{\beta+\gamma-1}u_n^3 d\tau < \infty,\end{aligned}\right\}\to(4.1)$$

and

$$\|u(t)\|_\gamma^2 - \|u(t_0)\|_\gamma^2 + 2v\int_{t_0}^{t}\|u\|_{\alpha+\gamma}^2 d\tau \ge 2\gamma\int_{t_0}^{t}\sum_{n=1}^{\infty}(n+1)^{\beta+\gamma-1}u_n^2 u_{n+1}d\tau \to (4.2)$$

for all $0 \le t_0 \le t$, $0 < \gamma \le 1$.

Proof:

By theorem (4.1.1), $u_n(t) \ge 0$ for all n, t > 0. Since $\|u(t)\|^3_{2(\beta+\gamma-1)/3}$ is integrable on [t_0, t] for all $0 \le t_0 \le t$, we obtain

$$\int_{t_0}^{t}\sum_{n=1}^{\infty}n^{\beta+\gamma-1}u_n^2 u_{n+1}d\tau \le 2\int_{t_0}^{t}\sum_{n=1}^{\infty}n^{\beta+\gamma-1}u_n^3 d\tau \le 2\int_{t_0}^{t}\left(\sum_{n=1}^{\infty}n^{2(\beta+\gamma-1)/3}u_n^2\right)^{3/2}d\tau = 2\int_{t_0}^{t}\|u\|^3_{2(\beta+\gamma-1)/3}d\tau < \infty$$

Hence, the relation in (4.1) hold. In particular,

$$\liminf_{n\to\infty}\int_{t_0}^{t}\sum_{n=1}^{\infty}n^{\beta+\gamma-1}u_n^2 u_{n+1}d\tau = 0 \qquad \to(4.3)$$

Now, multiplying by $n^\gamma u_n$,taking a summation from 1to N, and integrating from t_0 to t, we get

$$\sum_{n=1}^{N}n^\gamma u_n(t)^2 - \sum_{n=1}^{N}n^\gamma u_n(0)^2 + 2v\int_{t_0}^{t}\sum_{n=1}^{N}n^{\alpha+\gamma}u_n^2 d\tau = 2\int_{t_0}^{t}\sum_{n=1}^{N-1}(n+1)^\beta\left((n+1)^\gamma - n^\gamma\right)u_n^2 u_{n+1}d\tau - 2\int_{t_0}^{t}(N+1)^\beta N^\gamma u_N^2 u_{N+1}d\tau + 2\int_{t_0}^{t}\sum_{n=1}^{N}n^\gamma g_n u_n d\tau$$

$$\ge 2\gamma\int_{t_0}^{t}\sum_{n=1}^{N-1}(n+1)^{\beta+\gamma-1}u_n^2 u_{n+1}d\tau - 2\int_{t_0}^{t}(N+1)^\beta N^\gamma u_N^2 u_{N+1}d\tau$$

Taking the lower limit as $N \to \infty$, we get (4.2).

4.3 Non regular Weak Global Attractor

The weak global attractor for this system is

$$\mathcal{A}_w = \begin{Bmatrix} u_0 \in H: \exists\, a\ Leary-Hopf\ solution \\ u(t)\ on\ (-\infty,\infty), such\ that\ u(0) = u_0 \\ and\ |u(t)|\ is\ bounded\ on\ (-\infty,\infty) \end{Bmatrix}$$

4.3.1 Theorem

If $g_n = 0$ for all $n \geq N_g$, then every $u = (u_1, u_2, u_3, \dots\dots\dots) \in \mathcal{A}_w$ satisfies

$$u_n \geq 0 \quad n = 1,2,3 \dots\dots$$

Proof:

A general solution for $u_n(t)$ can be written as,

$$u_n(t) = u_n(t_0)\exp\left(-\int_{t_0}^{t}\left[vn^{\alpha} + (n+1)^{\beta}u_{n+1}(\tau)\right]d\tau\right) + \int_{t_0}^{t}\left(g_n + n^{\beta}u_{n-1}^2(s)\right) \exp\left(-\int_{s}^{t}\left[vn^{\alpha} + (n+1)^{\beta}u_{n+1}(\tau)\right]d\tau\right)ds \rightarrow (4.4$$

Clearly, this implies the following facts

- If $u_n(t_0) \geq 0$ for some n and t_0, then $u_n(t) \geq 0$ $\forall\, t \geq t_0$.
- If $|u(t)|$ is bounded $\forall\, t \in \mathcal{R}$, then $u_n(t) \geq 0\ \forall$ t $\geq \mathcal{R}$, whenever $u_{n+1}(\tau) \geq 0\ \forall\ t \in \mathcal{R}$.

Assume that there exist $u^0 \in \mathcal{A}_w, such\ that\ u_N^0 < 0\ \forall\, N \geq N_g$ then there exist a Leary-Hopf solution u(t), such that u(0) = u^0 and $|u(t)|$ is bounded on $(-\infty,\infty)$.For such a solution we have $u_N(t) < 0\ \forall\ t \leq 0$.From the energy inequality for u(t) we deduce that

$$\sum_{n=N}^{\infty}u_n(t_1)^2 - \sum_{n=N}^{\infty}u_n(t_0)^2 \leq 2\int_{t_0}^{t_1}\left[N^{\beta}u_{N-1}(\tau)^2u_N(\tau) - v\sum_{n=N}^{\infty}n^{\alpha}u_n(\tau)^2\right]d\tau \leq -2v\int_{t_0}^{t_1}\sum_{n=N}^{\infty}u_n(\tau)^2 d\tau$$

for all $t_0 \leq t_1 \leq 0$. Hence,

$$\sum_{n=N}^{\infty}u_n(t_0)^2 \geq e^{-2vt_0}\sum_{n=N}^{\infty}u_n(0)^2 \quad \text{for all } t_0 \leq 0.$$

This implies that $|u(t)|^2$ is not bounded. Which is contradiction to our assumption. Now take any $u^0 \in \mathcal{A}_w$. Then there exist a Leary-Hopf solution u(t), such that u(0) = u^0 and $|u(t)|$ is bounded on $(-\infty,\infty)$.For such a solution we proved that

$u_n(t) \geq 0\ \forall\ t \in \mathcal{R},\ N \geq N_g$.

If $u_{n+1} \geq 0$ for all t $\in \mathcal{R}$ then $u_n(t) \geq 0\ \forall\ t \in \mathcal{R}$.

4.4 Tridiagonal models for the Euler equations

4.4.1 Theorem

Let u(t) be a solution of (1) on [0, ∞) with v = 0, $g_n \geq 0$, $u_n(0) \geq 0$ for all n, and u(0) ≠ 0. Then $\|u(t)\|_{\frac{2(\beta+\gamma-1)}{3}}$ is not bounded on [0, ∞) for every $\gamma > 0$.

Proof:

Clearly, it is enough to prove the theorem in the case $0 < \gamma < \min\{1, 2(\beta-1)\}$. Assume that $\|u(t)\|_{\frac{2(\beta+\gamma-1)}{3}}$ is bounded on [0, ∞).

Then lemma (4.2.1)

$$\|u(t)\|_{\gamma}^2 - \|u(t_0)\|_{\gamma}^2 \geq 2\gamma\int_{t_0}^{t}\sum_{n=1}^{\infty}(n+1)^{\beta+\gamma-1}u_n^2u_{n+1}d\tau \geq 0 \rightarrow (4.5)$$

for all $0 \leq t_0 \leq t$.Thus, $\|u(t)\|_{\gamma}^2$ is non decreasing. Since $\gamma < \frac{2(\beta+\gamma-1)}{3}$, $\|u(t)\|_{\gamma}$ is bounded on [0, ∞). Then there exist $E_0 > 0$ such that

$\lim_{t\to\infty}\|u(t)\|_{\gamma}^2 = E_0$.

Then (4.5) implies that $\lim_{t\to\infty}\int_{t}^{\infty}u_n(\tau)^2u_{n+1}(\tau)d\tau = 0$,$n \in \mathcal{N}$ →(4.6)

Hence,

$$u_n(t)^2 - u_n(0)^2 = 2n^{\beta}\int_0^t u_{n-1}^2u_n\,d\tau - 2(n+1)^{\beta}\int_0^t u_n^2u_{n+1}d\tau + 2\int_0^t g_nu_n\,d\tau = 2n^{\beta}\int_0^{\infty}u_{n-1}^2u_n\,d\tau - 2(n+1)^{\beta}\int_0^{\infty}u_n^2u_{n+1}d\tau + 2\int_0^{\infty}g_nu_n\,d\tau$$

as $t \to \infty$.

Hence $u_n(\infty) = \lim_{t\to\infty}u_n(t)$ exists for all n. Now (4.6) implies that $u_n(\infty) \times u_{n+1}(\infty) = 0$ for all n. Suppose that $u_k(\infty) \neq 0$ for some k. Then $u_{k+1}(\infty) = 0$ and there exist $t_0 > 0$ such that

$$(k+2)^{\beta}u_{k+1}(t)u_{k+2}(t) \leq \tfrac{1}{3}(k+1)^{\beta}u_k(\infty)^2$$

and $$u_k^2(t) \geq \tfrac{2}{3}u_k(\infty)^2 \quad \forall\, t \geq t_0.$$

Thus,

$$\frac{d}{dt}u_{k+1} = (k+1)^{\beta}u_k(t)^2 - (k+2)^{\beta}u_{k+1}(t)u_{k+2}(t) + g_{k+1} \geq \tfrac{1}{3}(k+1)^{\beta}u_k(\infty)^2 \quad \forall\, n \geq t_0.$$

Therefore, $\lim_{t\to\infty}u_{k+1}(t) = \infty$ .Which is contradiction

5. The Dyadic Navier-Stokes Equations with Hyper-dissipation

Consider the dyadic Navier-Stokes equations with hyper-dissipation,

$$\frac{du}{dt} + C(u,u) + (\Delta)^{\alpha}u = 0.$$

Since

$$\langle C(u,u), u\rangle = 0,$$

We have

$$\frac{1}{2}\frac{d}{dt}(\langle u,u\rangle) + \langle(\Delta)^{\alpha}u, u\rangle = 0,$$

And therefore, we have the energy decay

$$\langle u,u\rangle + 2\int_0^t\langle(\Delta)^{\alpha}u,u\rangle = \langle u_0, u_0\rangle, \quad \rightarrow (5.1)$$

where $u_0 = u(x,\ 0)$.

.5.1 Lemma

For any $\in > 0$, there is $\delta(\in) > 0$, so that if we know that

$$E_{C(Q)} + W_{C(Q),t_0} > (1-\delta)\, 2^{-(4+\in)j(Q)}, \quad \rightarrow (5.2)$$

then there exist $Q' \in C(Q)$, so that

$$E_{Q'} + W_{Q',t_0} \geq 2^{-(4+\in)j(Q')}.$$

Proof:

Assume, $E_{Q'} + W_{Q',t_0} < 2^{-(4+\in)j(Q')} \rightarrow (5.3)$

$\forall\, Q' \in C(Q)$.

On the other hand, we have

$$E_{C(Q)} + W_{C(Q),t_0} = \sum_{Q' \in C(Q)} \left[E_{Q'} + W_{Q',t_0} \right] = \sum_{k=1}^{\infty} \sum_{Q' \in C^k(Q)} \left[E_{Q'} + W_{Q',t_0} \right] \quad \rightarrow (5.4)$$

Now because are in 3 dimension, there are exactly 2^{3k} elements $Q' \in C^k(Q)$ with $j(Q') = j(Q) + k$. $\rightarrow (5.5)$

Using (5.3) and (5.5) in (5.4) we obtain

$$E_{C(Q)} + W_{C(Q),t_0} < \sum_{k=1}^{\infty} 2^{3k}\, 2^{-(4+\in)(j(Q)+k)}$$

Which is same as

$$E_{C(Q)} + W_{C(Q),t_0} < 2^{-(4+\in)j(Q)} \sum_{k=1}^{\infty} 2^{-(1+\in)k} \quad \rightarrow (5.6)$$

Since

$$\sum_{k=1}^{\infty} 2^{-(1+\in)k} < 1.$$

Choose $0 < \delta < 1$ such that

$$\sum_{k=1}^{\infty} 2^{-(1+\in)k} < 1-\delta \quad \rightarrow (5.7)$$

Substitute (5.7) in (5.6) we get

$$E_{C(Q)} + W_{C(Q),t_0} < (1-\delta)\, 2^{-(4+\in)j(Q)}$$

Which contradicts the assumption of our lemma..

5.2 Lemma

Fix j_0 sufficiently large. Then there exist an $\in$, $0 < \in < 1 - 4\alpha$, so that if at a time t_0, we have

$$E_Q + W_{Q,t_0} \geq 2^{-(4+\in)j(Q)} \quad \rightarrow (5.8)$$

with j(Q) > j_0, then there is some t with t < $t_0 + T$, where

$$2^{\frac{(\in-1)j(Q)}{2}} < T < 2^{-2\alpha j},$$

and a cube $Q' \in C(Q)$ so that at a time t we have

$$E_{Q'} + W_{Q',t_0} \geq 2^{-(4+\in)j(Q')}.$$

6. CONCLUSION

In this dissertation we have discussed about the evolutionary system, and dyadic Navier-Stokes equation with hyper-dissipation. All solutions are strong or weak in global attractor for three dimensional Navier-Stokes equation..

7. REFERENCES

[1] Cheskidov A, Foias C (2006) On global attractors of the 3D Navier-Stokes equations, *J. Differential Equations*, 231: 714-754.

[2] Cheskidov A, Blow-up in finite time for the dyadic model of the Navier-Stokes equations, *Trans. Amer. Math. Soc.,* in press.

[3] Foias C, Manley OP, Rosa R, Teman R. Navier-Stokes Equation and Turbulence, Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 83, 2001.

[4] Friedlander S, Pavlovic N (2004) Blowup in a three-dimensional vector model for the Euler equations, *Comm. Pure Appl. Math.* 57(5): 705-725

[5] Kukavica I (2006) Role of the pressure for the validity of the energy equality for solutions of the Navier-Stokes equation, *J. Dynam. Differential Equations*, 18: 461-482.

[6] Kartz NH, Pavlovic N (2005) Finite time blow-up for a dyadic model of the Euler equations, *Trans. Amer. Math. Soc.* 357: 695-708.

[7] Sell GR, You Y, Dynamic of Evolutionary Equations, Appl. Math. Sci., vol. 143, Springer-Verlag, New York, 2002.

[8] Cannone M: Harmonic analysis tools for solving the incompressible Navier-Stokes equations, TO appear in Harmonic of Mathematical Fluid Dynamics 3 (2004).

[9] Katz NH, Pavlovic N, A cheap Caffarelli-Kohn-Nirenberg inequality for the Navier-Stokes Equation with hyper-dissipation, GAFA 12 (2002)

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