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Strong solutions under Navier stokes system in fluid mechanics

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Abstract

The study of the nonhomogeneous incompressible Navier-stokes system in dimension d ≥3.we use new a priori estimates, that enable us todeal with low-regularity data and vanishing density. In particular, we prove new well-posedness results that improve the results of Danchin[3] by considering a less regular initial density, without a lower bound. Also, we obtain the first uniqueness criterion for weak solutions which is at the scaling of the equation. We are ready to derive the higher order derivatives estimates of the density and velocity. Finally,we establish some a priori time weighted estimates independent of time interval. The idea is based on the parabolic property of the system.

Keywords Strong solution, Navier-Stokes system, fourier transform, Gagliardo-inequality and Poincare's inequality.

INTRODUCTION

The Cauchy problem for the non-homogeneous Navier-stokes system set in the whole space $\mathbb{R}^d$ with d ≥3.It reads

$$\begin{cases} \partial_t \rho + u.\nabla\rho = 0 \\ \rho\partial_t u + \rho u.\nabla u - \Delta u = -\nabla p \\ div = 0 \\ (\rho, u)_{t=0} = (\rho_0, u_0). \end{cases} \rightarrow (1.1)$$

Where the unknowns ρ, u and p stand for the density, velocity and pressure of the fluid.;they aare respectively $\mathbb{R}^+$, $\mathbb{R}^d$,and $\mathbb{R}$ valued and they are all functions of the space variable x and of the time variable t. the fluid is furthermore supposed to incompressible, hence the zero divergence condion on u. finally, the notation u. ∇ corresponds to the operator $\sum_{i=1}^{d} u^i \partial_i$.

At least formally, this equation verify the following conservation law:

$$\left\|\sqrt{\rho(t)}u(t)\right\|_2^2 + \int_0^t \|\nabla u(s)\|_2^2 ds = \left\|\sqrt{\rho_0}u_0\right\|_2^2$$

There is another conserved quality, namely for any $\alpha$and β the Lebesgue measure $\mu\left\{x \in \mathbb{R}^d, \alpha \le \rho(x,t) \le \beta\right\}$ is independent of t.

In particularthe L^∞ norm of the density as well as its lower bound, are independent of t. another important feature is the scaling invariance: if (u, ρ) is a solution associated to the initial data (u_0, ρ_0) ,then $\left(\lambda u\left(\lambda^2 t, \lambda x\right), \rho\left(\lambda^2 t, \lambda x\right)\right)$ will associated to $\left(\lambda u_0(\lambda x), \rho_0(\lambda x)\right)$. A functional space for the data (u_0, ρ_0)or for the solution $(u, \rho)$is said to be at the scaling of the equation if its norm is invariant by the above transformation.

The Navier-stokes equations are usually used to describe the motion of fluids. In particular, for the study of multiphase fluids without surface tension, the following density-dependent Navierstokes equations acts as model on some bounded domain $\Omega \subset R^N$ (N=2,3)

$$\begin{cases} \rho_t + div(\rho u) = 0 \;\; in\; \Omega \times (0,T] \\ (\rho u)_t + div(\rho u \otimes u) - div(2\mu(\rho)d) + \nabla p = 0, in\, \Omega \times (0,T], \\ div u = 0, \; in\, \Omega \times (0,T], \\ u = 0, on\, \Omega \times (0,T], \\ \rho_t = \rho_0\; u_t = 0\; in\, \Omega. \end{cases} \rightarrow (1.1.1)$$

Here ρ , u, and p denote the density, velocity and pressure of the fluid, respectively

Finally, we come to the most general case: velocity$\mu(\rho)$depends on density ρ

2. Strong Solutions

2.1 Definition

A solution (u, ρ) of (1.1) is said to be a finite energy solution if

$$\rho \in L^\infty\left([O,T], L^\infty\right), \sqrt{\rho}u \in L^\infty\left([O,T], L^2\right)$$

And $\nabla u \in L^2\left([O,T], L^2\right)$

2.2 Theorem (Coifman Rochberg Weiss)

If we denote by ϕ the multiplication operator by the function ϕ , and p is a calderon-zygmund operator, then the norm of the commutator is bounded as a linear operator on L^r.

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$$\left\|[\Phi, P]\right\|_{L(L^r)} \leq C_r \|\Phi\|_{BMO}$$

For some constant C_r .

2.3 Theorem

Let P verify

$$Q\left(\Phi\nabla_p\right) = f \text{ ,}$$

Where Φ is a positive function, then there holds

$$\left\|\nabla_p\right\|_2 \leq \left(\inf_x \Phi(x)\right)^{-1} \|f\|_2 .$$

And for $r \in (1, \infty)$, and for a constant C depending on r,

$$\left(\inf_x \Phi(x) - c\|\Phi\|_{BMO}\right)\left\|\nabla_p\right\|_r \leq \|f_r\|.$$

Proof:

Taking the scalar product of the equation with ∇_p .

$$Q.Q\left(\Phi\nabla_P\right) = f$$

$$\Phi\nabla_P = f$$

We can write

$$\Phi\nabla_P = [Q, \Phi]\nabla_p - Q\left(\Phi\nabla_P\right)$$

$$\Phi\nabla_P = [Q, \Phi]\nabla_p - f$$

Which implies, due to the result 1

$$\left\|\nabla_p\right\|_2 \leq \left(\inf_x \Phi(x)\right)^{-1} \|f\|_2$$

$$\left(\inf_x \Phi(x)\right)\left\|\nabla_p\right\|_2 \leq \|f\|_2$$

$$\left(\inf_x \Phi(x)\right)\left\|\nabla_p\right\|_r \leq \left\|\Phi\nabla_p\right\|_r$$

$$\leq \|f\|_r + C\|\Phi\|_{BMO}\left\|\nabla_p\right\|_r$$

$$\left(\inf_x \Phi(x)\right)\left\|\nabla_p\right\|_r \leq \|f\|_r + C\|\Phi\|_{BMO}\left\|\nabla_p\right\|_r$$

$$\left(\left(\inf_x \Phi(x)\right) - C\|\Phi\|_{BMO}\right)\left\|\nabla_p\right\|_r \leq \|f\|_r .$$

A Generalized Gronwall inequality

2.4 Theorem

Suppose that the following inequality holds

$$f' + (g')^2 \leq \alpha f + \beta g' g \qquad \rightarrow (2.1)$$

Where f, g', α and β are positive functions of the real variable such that

$f \in L^\infty, g(0) = 0,\ g' \in L^2, \alpha \in L'$ and $\sqrt{t}\beta(t) \in L^2$

Then there holds

Proof:

Let us consider

$$\int_0^t e^{-\int_0^s \alpha} g'(g' - \beta g)\, ds = f(t) e^{-\int_0^t \alpha} - f(0).$$

$$\alpha \in L', \sqrt{s}\beta \in L^2$$

Since $g(0) = 0$ .The holder's inequality gives that

$$g(s) \leq \sqrt{s}\sqrt{\int_0^s (g')^2} \qquad \rightarrow (2.2)$$

$$\left(e^{-\int_0^t \alpha} - \frac{1}{2} - \frac{1}{2}\int_0^t s\beta(s)^2\, ds\right)\int_0^t (g')^2$$

$$+ e^{-\int_0^t \alpha} f(t) \leq f(0)$$

$$\alpha \in L', \sqrt{s}\beta \in L^2$$

Since $g(0) = 0$ .The holder's inequality gives that

$$g(s) \leq \sqrt{s}\sqrt{\int_0^s (g')^2} \qquad \rightarrow (2.2)$$

We have

$$\int_0^t (g')^2 \leq e^{\int_0^t \alpha} \int_0^t e^{-\int_0^s \alpha} (g')^2\, ds$$

$$\leq e^{\int_0^t \alpha} \int_0^t e^{-\int_0^s \alpha} \left(g'^2 - g'\beta g + g'\beta g\right) ds$$

$$\leq e^{\int_0^t \alpha} \int_0^t e^{-\int_0^s \alpha} g'(g' - \beta g)\, ds$$

$$+ e^{\int_0^t \alpha} \int_0^t e^{-\int_0^s \alpha} \beta g g'\, ds$$

$$\leq e^{\int_0^t \alpha} \int_0^t e^{-\int_0^s \alpha} g'(g' - \beta g)\, ds + e^{\int_0^t \alpha} \int_0^t \left|\frac{g}{\sqrt{s}} \sqrt{s}\beta g'\right| ds$$

$$\leq e^{\int_0^t \alpha} \int_0^t e^{-\int_0^s \alpha} g'(g' - \beta g)\, ds + \frac{1}{2} e^{\int_0^t \alpha} \int_0^t (g')^2$$

$$+ \frac{1}{2} e^{\int_0^t \alpha} \left(\int_0^t \sqrt{s}\beta g'\, ds\right)^2$$

$$e^{-\int_0^t \alpha} \int_0^t (g')^2 \le \int_0^t e^{-\int_0^s \alpha} g'(g'-\beta g)\,ds + \frac{1}{2}\int_0^t (g')^2 + \frac{1}{2}\int_0^t s\beta^2\,ds \int_0^t (g')^2$$

$$\left(e^{-\int_0^t \alpha} - \frac{1}{2} - \frac{1}{2}\int_0^t s\beta^2\,ds\right)\int_0^t (g')^2 \le e^{-\int_0^s \alpha} g'(g'-\beta g)$$

We have the fundamental theorem of calculus,

$$f(t)e^{-\int_0^t \alpha} - f(0) \le \int_0^t e^{-\int_0^s \alpha} (f' - \alpha f)\,ds.$$

The equation

$$\int_0^t e^{-\int_0^s \alpha} g'(g'-\beta g)\,ds = f(t)e^{-\int_0^t \alpha} - f(0).$$

Using (2.1), we find the desired inequality

$$\left(e^{-\int_0^t \alpha} - \frac{1}{2} - \frac{1}{2}\int_0^t s\beta^2\,ds\right)\int_0^t (g')^2 + e^{-\int_0^t \alpha} f(t) \le f(0)$$

Some Commutator

We denote here Λ^σ for the Fourier multiplier defined in $F(\Lambda^\sigma f)(\varepsilon) = |\varepsilon|^\sigma \hat{f}(\varepsilon)$, and $[\Lambda^\sigma, f]$ for the commutator

$$[\Lambda^\sigma, f]g = \Lambda^\sigma(fg) - f\Lambda^\sigma g.$$

2.4 Theorem

Let d=3, $s \in \left(2, \frac{5}{2}\right)$, or $d \ge 4$, $s \in \left(\frac{d}{2}, \frac{d}{2}+1\right)$

and let u_0 and r_0 be functions such that

$$u_0 \in H^s, \frac{1}{r_0} \in L^\infty \text{ and } \Lambda^{s-1} r_0 \in L^{\frac{d}{s-1}}.$$

Suppose also that, for constant C,

$$\left\|\Lambda^{s-1} r_0\right\|_{\frac{d}{s-1}} \le C(\inf r_0).$$

Then there exists $T > 0$ and a local solution (u, r) of NNS defined on $[0,T]$ such that

$$u \in L^\infty([0,T], H^S) \cap L^2([0,T], H^{S+1})$$

$$\frac{1}{r} \in L^\infty([0,T], L^\infty) \text{ and}$$

$$\Lambda^{s-1} r \in L^\infty\left([0,T], L^{\frac{d}{s-1}}\right)$$

Proof:

We denote Λ^s for the Fourier multiplier defined in

$$f(\Lambda^\sigma f)(\varepsilon) = |\varepsilon|^\sigma \hat{f}(\varepsilon)$$

The homogenous norms $\|u\|_{H^S}$ and $\|\nabla p\|_{H^{S-1}}$.

Proof of the Pressure

Consider

$$r\Lambda^{s-1}\nabla p = \Lambda^{s-1}(r\nabla p) + \left[r, \Lambda^{s-1}\right]\nabla p$$

Taking the scalar product with $\Lambda^{s-1}\nabla p$.

$$(\inf r)\left\|\Lambda^{s-1}\nabla p\right\|_2 = \left\|Q\Lambda^{s-1}(r\nabla p)\right\|_2 + \left[r, \Lambda^{s-1}\right]\nabla p \to (2.3).$$

We have, due to the zero divergence of u

$$Q\Lambda^{S-1}(r\nabla_p) = -Q\Lambda^{S-1}(u.\nabla u) + Q\Lambda^{S-1}(r\nabla u)$$

Since H^S is algebra for $S > \frac{d}{2}$,

We find $\left\|\Lambda^{s-1}(u.\nabla u)\right\|_2 \le c\|u\|_{H^s}^2 \to (2.4)$

The theorem (2.2), the boundedness of Q on Lebesgue spaces.

$$\left\|Q\Lambda^{S-1}(r\Delta u)\right\|_2 \le \left\|[Q,r]\Lambda^{s-1}\Delta u\right\|_2 + \left\|Q\left[r,\Lambda^{s-1}\right]\Delta u\right\|_2.$$

$$\le C\|r\|_{BMO}\|u\|_{H^{S+1}} + C\left\|\Lambda^{s-1} r\right\|_{\frac{d}{s-1}}\|u\|_{H^{S+1}}$$

(∵ theorem 2.2)

$$\le C\left\|\Lambda^{s-1} r\right\|_{\frac{d}{s-1}}\|u\|_{H^{s+1}} \to (2.5)$$

The equation (2.4) and (2.5) substation in (2.3)

$$\left(\inf r - c\left\|\Lambda^{s-1} r\right\|_{\frac{d}{s-1}}\right)\left\|\Lambda^{s-1}\nabla p\right\|_2 \le C\|u\|_{H^s}^2 + C\left\|\Lambda^{s-1} r\right\|_{\frac{d}{s-1}}\|u\|_{H^{s+1}}$$

Proof of the Velocity

Consider the operator Λ^s to the momentum conservation equation in NNS.

Taking the scalar product with $\Lambda^s u$.

$$\frac{d}{dt}\left\|\Lambda^s u\right\|_2^2 = -\left\langle\Lambda^s(u,\nabla u), \Lambda^s u\right\rangle + \left\langle\Lambda^s(r\nabla u), \Lambda^s u\right\rangle - \left\langle\Lambda^s(r\nabla p), \Lambda^s u\right\rangle.$$

$= I + II + III$

identity

$$I = \left|\left\langle \Lambda^S (u.\nabla u), \Lambda^S u \right\rangle\right|$$

$$\leq \left|\left\langle \Lambda^S (u.\nabla u) - u.\nabla \Lambda^S u; \Lambda^S u \right\rangle\right|$$

$$\leq C \|\nabla u\|_\infty \|u\|_{H^S}^2 \leq C \|u\|_{H^{S+1}} \|u\|_{H^S}^2 .$$

Let us consider II,

$$II = \left\langle \Lambda^{s-1}(r\Delta u), \Lambda^{s+1} u \right\rangle$$

$$= -\left\langle r\Lambda^{s+1} u, \Lambda^{s+1} u \right\rangle - \left\langle \left[r, \Lambda^{s-1} \right] \Delta u, \Lambda^{s+1} u \right\rangle$$

Using the holder inequality

$$II \leq -(\inf r) \left\| \Lambda^{s+1} u \right\|_2^2 + C \left\| \Lambda^{s-1} r \right\|_{\frac{d}{s-1}} \left\| \Lambda^{s+1} u \right\|_2^2$$

Let us consider III

Using the incompressibility of u

$$III = \left\langle \Lambda^{s-1}(r\nabla p), \Lambda^{s+1} u \right\rangle$$

$$= \left\langle r\Lambda^{s-1} \nabla p, \Lambda^{s+1} u \right\rangle + \left\langle \left[\Lambda^{s-1}, r \right] \nabla p, \Lambda^{s+1} u \right\rangle .$$

$$= \left\langle [p, r] \Lambda^{s-1} \nabla_p, \Lambda^{s+1} u \right\rangle + \left\langle \left[\Lambda^{s-1}, r \right] \nabla_p, \Lambda^{s+1} u \right\rangle .$$

Using theorem 2.2

$$III \leq C \left\| \Lambda^{s-1} r \right\|_{\frac{d}{s-1}} \left\| \Lambda^{s-1} \nabla p \right\|_2 \left\| \Lambda^{s+1} u \right\|_2$$

The equation I, II and III, we have

$$\frac{d}{dt} \left\| \Lambda^s u \right\|_2^2 + \left((\inf r) \left\| \Lambda^{s+1} u \right\|_2^2 - C \left\| \Lambda^{s-1} r \right\|_{\frac{d}{s-1}} \right) \left\| \Lambda^{s+1} u \right\|_2^2$$

$$\leq C \|u\|_{H^{s+1}} \|u\|_{H^s}^2 + C \left\| \Lambda^{s-1} r \right\|_{\frac{d}{s-1}} \left\| \Lambda^{s-1} \nabla p \right\|_2 \left\| \Lambda^{s+1} u \right\|_2 \to (2.7)$$

PROOF OF THE DENSITY

The conservation of mass equation in NNS can be written

$$\partial_t r + u.\nabla_r = 0$$

Consider Λ^{s-1} to this equation taking the scalar product with $\left| \Lambda^{s-1} r \right|^{\frac{d}{s}-2} \Lambda^{s-1} r$,we get

$$\frac{d}{dt} \left\| \Lambda^{s-1} r \right\|_{\frac{d}{s-1}} \leq \left\| \left[\Lambda^{s-1}, u \right] . \nabla r \right\|_{\frac{d}{s-1}}$$

$$\leq C \left\| \Lambda^{s-2} \nabla r \right\|_{\frac{d}{s-1}} \|u\|_{H^{\frac{d}{2}+1}} \to (2.8)$$

Local control of U, P and R

The non-homogeneous norms $\|u\|_{H^S}$ and $\|\nabla p\|_{H^{S-1}}$

Assume that the initial data satisfy

$$(\inf r) - C \left\| \Lambda^{s-1} r \right\|_{\frac{d}{s-1}} = 2C > 0.$$

The equation (2.8) gives for $t \leq 1$

$$\left\| \Lambda^{s-1} r \right\|_{\frac{d}{s-1}} = 2Ce^{c\int_0^t \|u\|_{H^{S+1}}^2}$$

So for some $\in > 0$, $\int_0^t \|u\|_{H^{k+1}}^2 \leq \in$,we have

$(\inf r) - C \left\| \Lambda^{s-1} r \right\|_{\frac{d}{s-1}} \geq C$ and $\left\| \Lambda^{s-1} r \right\|_{\frac{d}{s-1}} \leq C'$

These two inequalities hold,a priori estimates we get

$$\frac{d}{dt} \|u\|_{H^{s+1}}^2 + C' \|u\|_{H^{s+1}}^2 \leq C \|u\|_{H^s}^4 .$$

The local control of u in $L_t^\infty H_x^s \cap L_t^2 H_x^{s+1}$ .This implies that

$$t > 0, \quad \int_0^t \|u\|_{H^{k+1}}^2 \leq \in .$$

The hypothesis of existence of $T > 0$ such that on $[0, T]$ an a equation of solution $(u, p, \nabla p)$ is bounded in

$$\left(L^\infty \left([0,T], H^S \right) \cap L^2 \left([0,T], H^{S+1} \right) \right)$$

$$\times L^\infty \left([0,T], W^{\frac{S-1}{d/S-1}} \right) \times L^2 \left([0,T], H^{S-1} \right) .$$

3. Coifman and Meyer's generalized product operators and littlewoods paley theory

3.1 Coifman and Meyer's generalized product operators

Let us first introduce the following notation for the generalized product operators of Coifman and Meyer's [3]

$$T_m(f, g) = \int_{R^d} e^{ix.(\varepsilon+\eta)} m(\varepsilon+\eta) \hat{f}(\varepsilon) \hat{g}(\eta) \, d\varepsilon d\eta$$

We now state the fundamental theorem of Coifman and Meyer.If kernel m satisfies

$$\left| \partial_\varepsilon^\alpha \partial_\eta^\beta m(\varepsilon, \eta) \right| \leq \frac{C}{(|\varepsilon| + |\eta|)^{|\alpha|+|\beta|}} \to (3.1)$$

Then t_m bounded from $L^p \times L^q$ to L^r, with

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{r},\ 1<p\ ,q\leq\infty, 1\leq r<\infty.$$

2.0.2 Little Wood Paley Theory

Consider ψ a function supported in the annulus centered in 0, the small radius ¾ and big radius 8/3, such that $\sum_{j\in z}\psi\left(\frac{\varepsilon}{2^j}\right)=1$ for $\varepsilon\neq 0$

Then the Fourier multipliers Δ_j and S_j are defined by

$$\Delta_j=\psi\left(\frac{D}{2^j}\right)$$

$$S_N=1-\sum_{j\geq N+1}\psi\left(\frac{D}{2^j}\right)$$

Thus any distribution can be decomposed (modulo polynomials) into a sum of elementary elements that are localized in frequency

$$f=\sum_{j\in z}\Delta_j f \quad \text{or} \quad f=S_N f+\sum_{j\geq N+1}\Delta_j f.$$

The L^2 -Sobolev norms can be expressed using the little wood –Paley decomposition

$$\|f\|_{H^s}^2 \approx \sum_{j\in z}2^{2js}\left\|\Delta_j f\right\|_{L^2}^2$$

If $f=\sum_j f_j$ with supp $\hat{f}_j\subset B\left(0,c2^j\right)$ and $s>0$, we have

$$\|f\|_{H^s}^2\leq C\sum_{j\in z}2^{2js}\left\|f_j\right\|_{L^2}^2 \qquad \rightarrow(3.2)$$

For L^P spaces, we have

$$\|f\|_{L^P}^2\leq C\sum_{j\in z}\left\|\Delta_j f\right\|_p^2 \quad \text{if} \quad P<2$$

If $f=\sum_j f_j$, with supp $\hat{f}_j\subset B\left(0,C2^j\right)$ if $S>0$ and $P\geq 2$

$$\|f\|_{W^{s,p}}^2\leq C\sum_{j\in z}2^{2js}\left\|f_j\right\|_{L^2}^2 \qquad \rightarrow(3.3)$$

Finally, we have the Bernstein inequality

$$\left\|s_j f_j\right\|_{L^p}\leq C2^{jd\left(\frac{1}{q}-\frac{1}{p}\right)}\left\|s_j f_j\right\|_{L^q} \quad \rightarrow(3.4)$$

If $1\leq p\leq q\leq\infty$, And we the following estimates

$$\left\|\Lambda^s s_j f\right\|_{L^p}\leq C2^{js}\left\|s_j f\right\|_p$$

$$\left\|\Lambda^s\Delta_j f\right\|_{L^p}\leq C2^{js}\left\|\Delta_j f\right\|_p \qquad \rightarrow(3.5)$$

2.1 Theorem

The following estimates hold true for $1<p<\infty$ and

$$\left\|\left[\Lambda^\sigma, f\right]g\right\|_p$$

$$\leq C\left(\|\nabla f\|_\infty\left\|\Lambda^{\sigma-1}g\right\|_p+\left\|\Lambda^\sigma f\right\|_p\|g\|_\infty\right)$$

For $1\leq\sigma<\frac{d}{2}$,

$$\left\|\left[\Lambda^\sigma, f\right]g\right\|_2\leq C\left\|\Lambda^\sigma f\right\|_{\frac{d}{\sigma}}\left\|\Lambda^\sigma g\right\|_2 \rightarrow(3.6)$$

For $1\leq\sigma<\frac{d}{2}$,

$$\left\|\left[\Lambda^\sigma, f\right]g\right\|_{\frac{d}{\sigma}}$$

$$\leq C\left\|\Lambda^{\sigma-1}g\right\|_{\frac{d}{\sigma}}\left(\|\nabla f\|_\infty+\|f\|_{H^{\frac{d}{2}+1}}\right)\rightarrow(3.7)$$

Proof:

Consider Coifman and Meyer's equation

$$Tm(f,g)=\int_{R^d}e^{ix(\varepsilon+\eta)}m(\varepsilon,\eta)\hat{f}(\varepsilon)\hat{g}(\eta)d\varepsilon\, d\eta$$

$$Tm(f,g)=\left[\Lambda^\sigma, f\right]g$$

Where, $m(\varepsilon,\eta)=\left(|\varepsilon+\eta|^\sigma-|\eta|^\sigma\right)$

The regions where $|\varepsilon|<<|\eta|$, $|\varepsilon|\approx|\eta|$ and $|\varepsilon|>>|\eta|$ and will have to use different techniques for each of those regions.

Let us consider $|\varepsilon|<<|\eta|$, We have

$$m_1(\varepsilon,\eta)=\Phi_1\left(\frac{|\varepsilon|}{|\eta|}\right)m(\varepsilon,\eta).$$

A small computation shows that m_1 can be written as

$$m_1(\varepsilon,\eta)=\sum_{k=1}^{d}\mu(\varepsilon,\eta)\frac{\varepsilon^k+2\eta^k}{|\eta|}\varepsilon^k|\eta|^{\sigma-1}$$

Where μ is a homogenous function of order 0. Hence satisfying the equation

This is also the case for

$$\bar{\mu}(\varepsilon,\eta)=\mu(\varepsilon,\eta)\frac{\varepsilon^k+2\eta^k}{|\eta|}$$

So we have $T_{m_1}(f,g)=T_{\bar{\mu}}\left(\nabla f,\Lambda^{\sigma-1}g\right)$, Where $\bar{\mu}$ the estimates we have,

$$\left\|T_{m_1}(f,g)\right\|_2 = \left\|T_{\bar{\mu}}\left(\nabla f, \Lambda^{\sigma-1} g\right)\right\|_2$$

$$\le C\|\nabla f\|_d \left\|\Lambda^{\sigma-1} g\right\|_{\frac{2d}{d-2}} \le C\left\|\Lambda^{\sigma} f\right\|_{\frac{d}{\sigma}} \left\|\Lambda^{\sigma} g\right\|_2.$$

$$\left\|T_{m_1}(f,g)\right\|_2 \le C\left\|\Lambda^{\sigma} f\right\|_{\frac{d}{\sigma}} \left\|\Lambda^{\sigma} g\right\|_2 \rightarrow (3.8)$$

Let the symbol

$$m_2(\varepsilon,\eta) = |\eta|^{\sigma}\left(\Phi_2\left(\frac{|\varepsilon|}{|\eta|}\right) + \Phi_3\left(\frac{|\varepsilon|}{|\eta|}\right)\right)$$

Which corresponds to the second component of our commutator in regions $|\varepsilon| \square |\eta|$ and $|\varepsilon| << |\eta|$.

We know that $T_{m_2}(f,g)$ is nothing but the operator of $\Lambda^{\sigma} g$ by f, so by the classical theorem

We get

$$\left\|T_{m_2}(f,g)\right\|_2 \le C\|f\|_{BMO}\left\|\Lambda^{\sigma} g\right\|_2$$

$$\le C\left\|\Lambda^{\sigma} f\right\|_{\frac{d}{\sigma}} \left\|\Lambda^{\sigma} g\right\|_2$$

$$\left\|T_{m_2}(f,g)\right\|_2 \le C\left\|\Lambda^{\sigma} f\right\|_{\frac{d}{\sigma}} \left\|\Lambda^{\sigma} g\right\|_2 \rightarrow (3.9)$$

Let the symbol

$$m_3(\varepsilon,\eta) = |\varepsilon+\eta|^{\sigma}\,\Phi\left(\frac{|\varepsilon|}{|\eta|}\right)$$

It corresponds to the first component of our commutator in the region $|\varepsilon| \square |\eta|$.

We observe that $T_m(f,g)$ is of the form

$$\Lambda^{\sigma}\sum \Delta_j f \Delta_j g$$

We have

$$\left\|\Lambda^{\sigma}\sum \Delta_j f \Delta_j g\right\|_2^2 = \left\|\sum \Delta_j f \Delta_j g\right\|_{H^{\sigma}}^2$$

$$\le C\sum_j 2^{2j\sigma}\left\|\Delta_j f \Delta_j g\right\|_2^2$$

$$\le C\sum_j 2^{2j\sigma}\left\|\Delta_j f \Delta_j g\right\|_2^2$$

$$\le C\sum_j 2^{2j\sigma}\left\|\Delta_j f\right\|_{\infty}\left\|\Delta_j g\right\|_2^2$$

$$\le C\left\|\Delta_j f\right\|_{\infty}\sum_j 2^{2j\sigma}\left\|\Delta_j g\right\|_2^2$$

$$\le C\left\|\Lambda^{\sigma} f\right\|_{\frac{d}{\sigma}}^2\left\|\Lambda^{\sigma} g\right\|_2^2$$

By Bernstein's inequality (3.4)and (3.5)

The last consider the symbol

$$m_4(\varepsilon,\eta) = |\varepsilon+\eta|^{\sigma}\,\Phi_3\left(\frac{|\varepsilon|}{|\eta|}\right)$$

It is first component of our in commutator in the region $|\varepsilon| >> |\eta|$ We can write m_4

$$m_4(\varepsilon,\eta) = \Phi_3\left(\frac{|\varepsilon|}{|\eta|}\right)\frac{|\varepsilon+\eta|^{\sigma}}{|\varepsilon|^{\sigma}}\,|\varepsilon|^{\sigma} = \rho(\varepsilon)|\varepsilon|^{\sigma}$$

Applying the theorem of Coifman and Meyer we get

$$\left\|T_{m_4}(f,g)\right\|_2 = \left\|T_p\left(\Lambda^{\sigma} f, g\right)\right\|_2$$

$$\le C\left\|\Lambda^{\sigma} f\right\|_{\frac{d}{\sigma}}\|g\|_{\frac{2d}{d-2\sigma}}$$

$$\le C\left\|\Lambda^{\sigma} f\right\|_{\frac{d}{\sigma}}\left\|\Lambda^{\sigma} g\right\|_2$$

The conclude the proof of (3.6), to observe that

$$m = m_1 + m_2 + m_3 + m_4$$

$$\left[\Lambda^{\sigma}, f\right]g$$

$$= \int_{R^d} e^{ix(\varepsilon+\eta)}\left(|\varepsilon+\eta|^{\sigma} - |\eta|^{\sigma}\right)\hat{f}(\varepsilon)\hat{g}(\eta)\,d\varepsilon\,d\eta$$

To prove The equation (3.7)

We can write

We consider first

$$m_1(\varepsilon,\eta) = m(\varepsilon,\eta)\,\Phi\left(\frac{|\varepsilon|}{|\eta|}\right) = \bar{\mu}(\varepsilon,\eta)\,\varepsilon^k\,|\eta|^{\sigma-1}$$

The use theorem of Coifman and Meyer to bound

$$\left\|T_{m_1}(f,g)\right\|_{\frac{d}{\sigma}} = \left\|T_{\bar{\mu}}\left(\nabla f, \Lambda^{\sigma-1} g\right)\right\|_{\frac{d}{\sigma}}$$

$$\le C\|\nabla f\|_{\infty}\left\|\Lambda^{\sigma-1} g\right\|_{\frac{d}{\sigma}}$$

Let us consider the symbol

$$n_1(\varepsilon,\eta) = |\varepsilon+\eta|^{\sigma}\left(\Phi_2\left(\frac{|\varepsilon|}{|\eta|}\right) + \Phi_3\left(\frac{|\varepsilon|}{|\eta|}\right)\right)$$

Consider

$T_{n_1}(f,g)$ becomes equivalent to

$\Lambda^{\sigma}\sum_j \Delta_j f\, S_{j+3} g$, .Using (3.3)

$$\left\|\Lambda^{\sigma}\sum_j \Delta_j f\, S_{j+3} g\right\|_{\frac{d}{\sigma}}^2 \le C\sum_j 2^{2j\sigma}\left\|\Delta_j f\, S_{j+3} g\right\|_{\frac{d}{\sigma}}^2$$

$$\le C\sum_j 2^{2j\sigma}\left\|\Delta_j f\right\|_p^2 \left\|S_{j+3}\right\|_d^2$$

With $\dfrac{1}{p}+\dfrac{1}{d}=\dfrac{\sigma}{d}$

$$\le C\left\|g\right\|_d^2 \sum_j 2^{2j\sigma}\left\|\Delta_j f\right\|_p^2$$

$$\le C\left\|g\right\|_d^2 \sum_j 2^{j(d+2)}\left\|\Delta_j f\right\|_2^2$$

By Bernstein inequality

$$\le C\left\|g\right\|_d^2 \left\|\Lambda^{\left(\frac{d}{2}+1\right)} f\right\|_2^2 .$$

Consider the symbol

$$n_2(\varepsilon,\eta)=|\eta|^{\sigma}\left(\Phi_2\left(\frac{|\varepsilon|}{|\eta|}\right)+\Phi_3\left(\frac{|\varepsilon|}{|\eta|}\right)\right)$$

It corresponds to frequencies $|\varepsilon| \square |\eta|$ (or) $|\varepsilon| >> |\eta|$, so if we switch to Little wood-Paley theory, $T_{n_1}(f,g)$ becomes equivalent to

$$\sum_j \Delta_j f\, S_{j+3} g$$

By the Sobolev embedding theorem for small, one has

$$\left\|\sum_j \Delta_j f\, S_{j+3} g\right\|_{\frac{d}{\sigma}} \le \left\|\sum_j \Delta_j f\,\Lambda^{\sigma} S_{j+3} g\right\|_{w^{\in,q}} \text{ with}$$

$$\frac{d}{q}-\in = s$$

$$\left\|\sum_j \Delta_j f\,\Lambda^{\sigma} S_{j+3} g\right\| \le C\sum_j 2^{2j\in}\left\|\Delta_j f\, S_{j+3} g\right\|_q^2$$

With $\dfrac{1}{p}+\dfrac{1}{d}=\dfrac{1}{q}$

$$\le C\left\|g\right\|_d^2 \sum_j 2^{2j(\in+\sigma)}\left\|\Delta_j f\right\|_p^2$$

$$\le C\sum_j 2^{2j\in}\left\|\Delta_j f\right\|_p^2 \left\|\Lambda^{\sigma} S_{j+3} g\right\|_d^2$$

With $\dfrac{1}{p}+\dfrac{1}{d}=\dfrac{1}{q}$

$$\le C\left\|g\right\|_d^2 \sum_j 2^{2j(\in+\sigma)}\left\|\Delta_j f\right\|_p^2$$

$$\le C\left\|g\right\|_d^2 \sum_j 2^{j(d+2)}\left\|\Delta_j f\right\|_2^2$$

By Bernstein inequality

$$\left\|T_{n_2}(f,g)\right\|_{\frac{d}{\sigma}} \le C\left\|g\right\|_d^2 \left\|\Lambda^{\left(\frac{d}{2}+1\right)} f\right\|_2^2$$

We conclude that

$$m = m_1 + n_1 + n_2$$

Hence the result of (3.7).

4. Energy Level Estimates And Higher Order Level Estimates

4.1 Energy Level Estimates

First, as the density satisfies the transport equation $(1.1.1)_1$, and making use of $(1.1.1)_3$.

4.1..1 Theorem

Suppose (ρ,u,p) is a strong solution to (1.1.1) on $\left[0,T^*\right)$. Under the assumption $\sup_{0<t<T^*}\left\|\nabla\mu(\rho)(t)\right\|_{L^P} = M < +\infty$, it holds that for every $t\in\left[0,T^*\right)$

$$\left\|\nabla u\right\|_{H^1} \le C\left\|\rho u_t\right\|_{L^2} + C\left\|\rho u_t\right\|_{L^4}^2 \left\|\nabla u\right\|_{L^2}$$

And consequently by Sobolev embedding,

$$\left\|\nabla u\right\|_{H^1} \le C\left\|\rho u_t\right\|_{L^2} + C\left\|\nabla u\right\|_{L^2}^3$$

Proof:

Assume that $\left\|u\right\|_{H^2} \le C\left\|F\right\|_{L^2}\left(1+\left\|\nabla\mu(\rho)\right\|_{L^P}\right)^{\frac{P}{P-2}}$ and According to above equation Gagliardo-Nirenberg inequality

$$\left\|\nabla u\right\|_{H^1} \le C\left(\left\|\rho u_t\right\|_{L^2} + \left\|\rho u.\nabla u\right\|_{L^2}\right).$$

$$\left(1+\left\|\nabla\mu(\rho)\right\|_{L^P}\right)^{\frac{P}{P-2}}$$

$$\le C\left\|\rho u_t\right\|_{L^2} + C\left\|\rho u\right\|_{L^4}\left\|\nabla u\right\|_{L^2}^{\frac{1}{2}}\left\|\nabla u\right\|_{H^1}^{\frac{1}{2}}$$

$$\le C\left\|\rho u_t\right\|_{L^2} + C\left\|\rho u\right\|_{L^4}^2\left\|\nabla u\right\|_{L^2} + \frac{1}{2}\left\|\nabla u\right\|_{H^1}.$$

4.2.Higher Order Level Estimates

Now we are ready to derive the higher order derivatives estimates of the density of the velocity.

4.2.1 Theorem

Suppose (ρ, u, p) is a strong solution to (1.1.1) on $\left[0, T^*\right)$. Under the assumption

$$\sup_{0<t<T^*} \left\|\nabla \mu(\rho)(t)\right\|_{L^P} = M < +\infty,$$ there exists a generic positive constant C such that

$$\sup_{0<t<T^*} \left[\left\|\sqrt{\rho} u_t(t)\right\|_{L^2}^2 + \int_0^t \left\|\nabla u_t\right\|_{L^2}^2 ds\right] \le C.$$

5. A Priori Estimates

. In this section, the constant C will denote some positive constant which depends only on $\Omega, q, \bar{\rho}, \underline{\mu}, \bar{\mu}, \left\|\nabla u_0\right\|_{L^2}$ but independent of time T.

5.1. Theorem

Suppose (ρ, u, P) is the unique local strong solution to (1.1.1) on $[0, T]$, with the data (ρ_0, u_0), it holds that

$$\int \rho |u(t)|^2 dx + \int_0^t \int |\nabla u|^2 dx\, ds \le C \int \rho_0 |u_0|^2 dx \rightarrow (5.1)$$

,For every $t \in [0, T]$, furthermore,

$$\sup_{t\in[0,T]} t \left\|\sqrt{\rho} u(t)\right\|_{L^2}^2 + \int_0^T t \left\|\nabla u\right\|_{L^2}^2 dt \le C \int \rho_0 |u_0|^2 dx \qquad \rightarrow (5.2)$$

Proof:

The proof of (5.1), using the lemma

"Suppose (ρ, u, P) is a strong solution to (1.1.1) on $\left[0, T^*\right)$ .then for every $t \in \left[0, T^*\right)$,

$$\frac{1}{2} \int \rho |u(t)|^2 dx + 2 \int_0^t \int \mu(\rho) |d|^2 dx\, ds \le \frac{1}{2} \int \rho_0 |u_0|^2 dx \rightarrow (5.3)$$

Since $\mu(\rho) \ge \underline{\mu}$, and $2\int |d|^2 dx = \int |\nabla u|^2 dx$, owing to div=0, then (5.3) implies

$$\int_0^t \int |\nabla u|^2 dx\, ds \le C \int \rho_0 |u_0|^2 dx \qquad \rightarrow (5.4)."$$

We have the equation (5.3)

$$\frac{1}{2} \int \rho |u(t)|^2 dx + 2 \int_0^t \int \mu(\rho) |d|^2 dx\, ds \le \frac{1}{2} \int \rho_0 |u_0|^2 dx$$

Substitution in (5.4)

$$\int \rho |u(t)|^2 dx + \int_0^t \int |\nabla u|^2 dx\, ds \le C \int \rho_0 |u_0|^2 dx$$

Hence the proof of (5.1)

Next to proof (5.2)

Let

$$\frac{1}{2} \frac{d}{dt} \int \rho |u|^2 dx + 2 \int \mu(\rho) |d|^2 dx = 0 \rightarrow (5.5)$$

Since Ω is bounded domain, one can deduce from Poincare's inequality that

$$\frac{1}{2} \int \rho |u|^2 dx \le C \|u\|_{L^2}^2 \le C \|\nabla u\|_{L^2}^2 \le C \int \mu(\rho) |d|^2 dx$$

Where the fact $\mu(\rho) \ge \underline{\mu} > 0$ is used. Combining (5.5) and (5.6), we obtain

$$\int \rho |u(t)|^2 dx \le C e^{-ct} \int \rho_0 |u_0|^2 dx$$

Multiplying the equation (5.5) by t and integrating over Ω, one has

$$\frac{d}{dt} \int \frac{t}{2} \rho |u|^2 dx + 2t \int \mu(\rho) |d|^2 dx = \frac{1}{2} \int \rho |u|^2 dx,$$

Which implies that

$$\sup_{t\in[0,T]} t \left\|\sqrt{\rho} u(t)\right\|_{L^2}^2 + \int_0^T t \|\nabla u\|_{L^2}^2 dt \le C \int_0^T \int \rho |u|^2 dx dt \le C \int \rho_0 |u_0|^2 dx$$

$$\sup_{t\in[0,T]} t \left\|\sqrt{\rho} u(t)\right\|_{L^2}^2 + \int_0^T t \|\nabla u\|_{L^2}^2 dt \le C \int \rho_0 |u_0|^2 dx.$$

5.2. Theorem

Suppose (ρ, u, P) is the unique local strong solution to (1.1.1) on $[0, T]$ and satisfies

$$\sup_{t\in[0,T]} \left\|\nabla \mu(\rho(t))\right\|_{L^q} \le 1.$$

Then there exists a generic positive constant C independent of time T, such that

$$\|\nabla u\|_{L^1(0,T,L^\infty)} \le C.$$

6. CONCLUSION

In this paper, we discussed the Strong solutions of Navier-Stokes system.The important discussion in the dissertation was Coifman and Meyer's generalized product operators and Littlewoods Paley theory,and also discussed the Energy level estimates and higher order level estimates.Finally, we discussed the A priori estimates bring elaborated.

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