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Stability Theory Of Differential Equation to Power Control in Wireless Network

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Abstract

In this paper we focus on stability theory of differential equation, concentrating in particular on systems of first-order ordinary differential equations and delay differential equations. In doing this, we investigate in detail the methods of Lyapunov for ordinary differential equation systems and the extension of these due to Razuikhin or delay differential equation systems. We then investigate the application of some of these methods to an important general class of continuous-time algorithms for the control of antenna transmission powers in wireless networks.

Keywords Stability, stable, unstable, uniformly stable, asymptotically uniformly stable, globally asymptotically uniformly stable.

INTRODUCTION

Systems of first- order differential equations are ubiquitous throughout applied mathematics. The general system formulation can be written as

$$\dot{x} = \Delta(t, x),$$

Where Δ

Is an all-encompassing quantity taken to represent whatever dependence may be present on the time t and the N-dimensional state variable x. we shall review in detail the important approaches in this area pioneered by Lyapunov in the late eighteenth century, which have grown to become some of the most important ideas in modern system analysis .Finally these results are illustrated by means of numerical simulations for a prototype model of a wireless network whose users are in relative motion.

1.1 Definition

An equation involving derivatives or differentials of one or differentials of one or more dependent variables with respect to one or more independent variables is called a **differential equation**.

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Example

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - y = \sin x.$$

1.2 Definition

Delay differential equation is a type of a differential equation in which the derivative of the unknown function at previous times. Delay differential equation is also called as a time-delay system.

1.3 Definition

A solution $x = X(t)$ of $\dot{x}$=f (t, x) is said to be **stable** if, given any $\in > 0$ and any $t_0 \geq 0$, there exist a $\delta = \delta(\in, t_0)$ such that

$$|x(t_0) - X(t_0)| < \delta$$

$$|x(t) - X(t)| < \in, \quad \forall\, t \geq t_0 \geq 0$$

for any solution of $\dot{x}$=f (t, x).

1.4 Definition

Uniformly stable if, for every $\in > 0$, there exist, $\delta = \delta(\in)$

$\in$ independent to t_o such that

$$|x(t) - X(t)| < \in, \quad \forall\, t \geq t_0 \geq 0$$

is satisfies for all $t_0 \geq 0$.

1.5 Definition

Unstable if it is not stable is said to unstable.

1.6 Definition

Asymptotically stable if it is stable and for any $t_0 \geq 0$ there exist a positive constant $c = c(t_o)$ such that,

$$|x(t_o) - X(t_o)| < c$$
$$x(t) - X(t) \to 0$$

as $t \to \infty$ for any solution $x(t)$ of $\dot{x} = f(t, x)$.

1.7 Definition

Uniformly asymptotically stable if it is uniformly stable and there exist a positive constant c, independent of t_o, such that, for every $\eta > 0$, there exist $T = T(\eta) > 0$ such that, for all $t_o \geq 0$

$$|x(t_o) - X(t_0)| < \mathrm{c}$$
$$|x(t) - X(t)| < \eta, \quad \forall\, t \geq t_o + T(\eta),$$

for any solution $x(t)$ of $\dot{x} = f(t, x)$.

1.8Definition

Globally uniformly asymptotically stable if it is uniformly stable with $\delta(\epsilon)$ satisfying $\lim_{\epsilon \to \infty} \delta(\epsilon) = \infty$, and, for all positive η and c, there exist $T = T(\eta, c) > 0$ such that, for all $t_o \geq o$

$$|x(t_0) - X(t_o)| < c$$
$$|x(t) - X(t)| < \eta \ \ \forall t \geq t_o + T(\eta, C)$$

for any solution $x(t)$ of $\dot{x} = f(t, x)$.

2. STABILITY PROPERTIES OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS WITH RANDOM COEFFICIENTS

2.1 Theorem

If,

$$\lim_{n \to \infty} inf \sum_{k=}^{n} \ln \left\{\frac{d_k^2+1}{2} + \frac{|d_k^2+1|}{2} |\phi_k(2a_k)|\right\} = -\infty \left(d_k := \frac{a_k}{a_{k+1}}\right) \ldots \ldots (2.1.1)$$

then it is almost sure in the probability space (Ω, A, P) that

$$\lim_{t \to \infty} inf \left\{x^2(t) + \frac{(x'(t))^2}{a(t)}\right\} = 0 \quad \ldots \ldots (2.1.2)$$

For all solutions of equation

$$x'' + a^2(t)x = 0, \quad a(t) := a_k$$
$$if\ t_{k-1} \leq t < t_k \quad (k \in \mathbb{N}).$$

Proof:

Given

$$\lim_{n \to \infty} inf \sum_{k=}^{n} \ln \left\{\frac{d_k^2 + 1}{2} + \frac{|d_k^2 + 1|}{2} |\phi_k(2a_k)|\right\} = -\infty \left(d_k := \frac{a_k}{a_{k+1}}\right)$$

To prove:

It is almost sure in the probability space (Ω, A, P) that

$$\lim_{t \to \infty} inf \left\{x^2(t) + \frac{(x'(t))^2}{a(t)}\right\} = 0$$

For all solutions of equation

$$x'' + a^2(t)x = 0, \quad a(t) := a_k$$
$$if\ t_{k-1} \leq t < t_k \quad (k \in \mathbb{N}).$$

If in addition, there exists a $\delta > 0$ such that the distribution function F_k of $t_k - t_{k-1}$, satisfies the inequality

$$F_k(\Pi / a_k - 0) \leq 1 - \delta \ \ (k \epsilon \mathbb{N}) \qquad \ldots (2.1.3)$$

Then it is almost sure that the lower limits of amplitudes of all the solutions equal zero.

The Stability property (2.1.2) can also be formulated in the form

$P(\|z_k\| \leq \varepsilon\ infinitely\ often) = 1$ for every $\varepsilon > 0$ …… (2.1.4)

If(2.1.3) also holds, then we have

$$P(amplitudes\ of\ x \leq \varepsilon\ infinitely\ often) = 1\ for\ every\ \varepsilon > 0 \quad \ldots \ldots (2.1.5)$$

The right-hand side of (2.1.2) immediately yields the following theorem that provides a condition for the convergence in L^2 of $\{\|z_k\|\}$ to zero.

Which implies the convergence "in probability"

i.e. $\lim_{k \to \infty} P(\|z_k\| \leq \varepsilon) = 1$ for every $\varepsilon > 0$ …… (2.1.6)

Hence the proof.

2.2 Theorem

If

$$\sum_{k=1}^{\infty} \left[\frac{a_k}{a_{k+1}} - 1\right]_+ < \infty, \qquad \ldots \ldots (2.2.1)$$

and

$$\sum_{k=1}^{\infty} 1 - |\phi_k(2a_k)|\} \left[\frac{a_k}{a_{k+1}} - 1\right]_- = \infty \quad \ldots \ldots (2.2.2)$$

Then $P(\lim_{k \to \infty} \|z_k\| = 0) = 1$ holds for all solution of 2.1.3 consequently, for every solution x of 2.1.1 we have

$$\lim_{t \to \infty} x(t) = 0\ almost\ surely \ldots \ldots (2.2.3)$$

Proof:

Given

If

$$\sum_{k=1}^{\infty}\left[\frac{a_k}{a_{k+1}}-1\right]_+ < \infty,$$

And

$$\sum_{k=1}^{\infty} 1-|\phi_k(2a_k)|\}\left[\frac{a_k}{a_{k+1}}-1\right]_- = \infty$$

To Prove:

Then $P(\lim_{k\to\infty}\|z_k\| = 0) = 1$ holds for all solution of 2.1.3 consequently, for every solution x of 2.1.1 we have

$$\lim_{t\to\infty} x(t) = 0 \ \ almost\ surely.$$

Suppose that we have

$$\left(\prod_{n-1}^{k}(1-[d_n-1]_-)\right)\|z_o\| \le \|z_k\| \le \left(\prod_{n-1}^{k}(1+[d_n-1]_+)\right)\|z_0\|$$

Where

$$[d]_+ = max\{d;0\},$$
$$[d]_- = max\{-d;0\}\ \ d\epsilon\mathbb{R}$$

In general above equation cannot be used to estimate $\|z_k\|$ by means of $\{a_k\}$.

The estimate of above equation tell us nothing about the asymptotic behavior of $\{\|z_k\|\}$.

$$\prod_{k=1}^{\infty}(1-[d_k-1]_-) = 0 \qquad \ldots\ldots(2.2.4)$$

Is necessary for $\underline{l} = 0$ in the case of a non-trivial solution. According to the theorem 2.1

We know that

$$\lim_{n\to\infty} inf \sum_{k=1}^{n} \ln\left\{\frac{d_k^2+1}{2}+\frac{|d_k^2+1|}{2}|\Phi_k(2)a_k|\right\} = -\infty\left(d_k = \frac{a_k}{a_{k+1}}\right)$$

Should imply 2.2.4

In fact by the inequality between the arithmetic mean and geometric mean we have

$$\sum_{k=1}^{n}\ln\frac{d_k^2+1}{2} \ge \sum_{k=1}^{n}\ln d_k = \ln\prod_{k=1}^{n} d_k \ge \ln\prod_{k=1}^{n}(1-[d_k-1]_-)$$

Therefore 2.1.1 implies 2.2.4. In general, however, the converse is not true. Now we will turn to proof of theorem

For example, if

$$a_0 := 1\ a_{2k-1} = \frac{3^{k-1}}{2^k}\ \ a_{2k} = \frac{3^k}{2^k}\ \ k\epsilon\mathbb{N} \qquad \ldots\ldots(2.2.5)$$

Then 2.2.4 obviously holds, but

$$\sum_{n=1}^{\infty}\ln\left(\frac{1+d_n^2}{2}\right) = \sum_{k=1}^{\infty}\left(\ln\frac{5}{9}+\ln\frac{5}{2}\right) = \sum_{k=1}^{\infty}\ln\frac{25}{18} = \infty$$

Shows that 2.1.1 is not satisfied, independently of the characteristic function $\{\phi_k\}$. In the monotonous $\{a_k\}$ the necessary condition and the sufficient condition are closer each other.

By

$$\|\mathbb{Z}_{k-1}\| \ge \|\mathbb{Z}_k\| \ge \left(\prod_{n=1}^{k} d_n\right)\|\mathbb{Z}_0\| = \frac{a_1}{a_{k+1}}\|\mathbb{Z}_0\|\ \ (k \in \mathbb{N})\ \ if\{a_k\} is\ increasing,$$

$$\|\mathbb{Z}_{k-1}\| \le \|\mathbb{Z}_k\| \le \left(\prod_{n=1}^{k} d_n\right)\|\mathbb{Z}_0\| = \frac{a_1}{a_{k+1}}\|\mathbb{Z}_0\|\ \ (k \in \mathbb{N})\ \ if\{a_k\}\ is\ decreasing$$

if $\{a_k\}$ is increasing, then

$$\lim_{k\to\infty} a_k = \infty \qquad \ldots\ldots(2.2.6)$$

is necessary for any stability property.

On the other hand, since $d_k < 1$ and

$$\frac{a_1}{a_k} = \prod_{n=1}^{k-1} d_n$$

$$= \prod_{n=1}^{k-1}(1-[1-d_n])\quad k\epsilon\mathbb{N}$$

(2.2.6) is equivalent to

$$\sum_{n=1}^{\infty}(1-d_n]=\sum_{n=1}^{\infty}[d_n-1]_-=\infty.$$

So 2.2.6 is also sufficient for a wide class of distributions.

Let the co-efficient a^2 in be a step function having the property

$a(t_1)\leq a(t_2)$ $(t_1\leq t_2)$, $\lim_{t\to\infty}a(t)=\infty$ i.e.,

$$a(t)=a_k \quad t_{k-1}\leq t<t_k \quad k\epsilon\mathbb{N} \quad a_k\nearrow\infty(k\to\infty)$$

Suppose that the difference $\{t_k-t_{k-1}\}_{k=1}^{\infty}$ are independent identically distributed random variables, whose characteristic function is denoted by ϕ if

$$\lim_{|s|\to\infty}sup|\phi(s)|<1$$

Then for arbitrary $\{a_k\}_{k=1}^{\infty}$ we have

$$\lim_{t\to\infty}x(t)=0 \ almost\ surely$$

For every solution x of

$$x''+a^2(t)x=0 \quad (t\geq 0)$$

Now we will turn to the proof of theorem 2.2. From Lebesgue's Dominated convergence theorem ("Let $\{f_n\}$ be a sequence of measurable f_n on x and suppose that

a) $0\leq f_1(x)\leq f_2(x)\leq\cdots.\alpha$; for all $x\epsilon X$.

b) $f_n(x)\to f(x) as n\to\infty for\ all\ x\epsilon$

Then f is measurable and $\int_x f_n dy\to\int_x f dy$ ")and the expected value of the bounded random variable $l(z_0)$ can be estimated as follows

$$\varepsilon l(z_0)=\varepsilon(\lim_{k\to\infty}\|z_k\|^2)$$

$$=\int_{\Omega}^{-}(\lim_{k\to\infty}\|z_k\|^2)dp$$

$$=\lim_{k\to\infty}\int_{\Omega}\|z_k\|^2dp$$

$$=\lim_{k\to\infty}\varepsilon(\|z_k\|^2)\leq\left(\lim_{k\to\infty}inf\left(\prod_{i=1}^{k}\Lambda_i\right)\right)\|z_0\|^2$$

We will show that the "lim inf" in the last expression equals 0. Since later on we will need to make similar

$$\frac{d_k^2+1}{2}\pm\frac{|d_k^2+1|}{2}k_k=1+\frac{d_k^2+1}{2}\pm\frac{|d_k^2+1|}{2}k_k$$

$$=1+\frac{1\pm k_k}{2}[d_k^2-1]_+$$

$$-\frac{1\mp k_k}{2}[d_k^2-1]_- \quad \ldots\ldots(2.2.7)$$

According to lemma 3.1 it is enough to show that

$$\lim_{k\to\infty}inf\sum_{i=1}^{k}\frac{(d_i+1)(1+k_i)}{2}[d_i-1]_+-\frac{(d_i+1)(1+k_i)}{2}[d_i-1]_-\Big)=-\infty \ \ldots(2.2.8)$$

But $k_i\leq 1\ (i\epsilon\mathbb{N})$ and d_i is bounded. So

(2.2.2 ⇒ 2.2.8) So,

$$\sum_{k=1}^{\infty}\{1-|\Phi_k(2a_k)|\}$$

$$\Rightarrow\lim_{k\to\infty}inf\sum_{i=1}^{k}\frac{(d_i+1)(1+k_i)}{2}[d_i-1]_+-\frac{(d_i+1)(1-k_i)}{2}[d_i-1]_-$$

$$=\infty$$

Hence the proof.

3.STABILITY THEORY OF DIFFERENTIAL EQUATION TO POWER CONTROL IN WIRELESS NETWORK

3.1 Theorem

Let $x=0$ be an equilibrium of the system $\dot{x}=f(t,x)$ and $U\subset D$ be a domain containing $x=0$. Suppose that there exist a continuous function $V:[0,\infty)\times\mathbb{R}^N\to\mathbb{R}$ such that with the time –derivative along the system trajectories defined as

$$\dot{V}(t,x(t))=limsup_{h\to 0+}\frac{1}{h}\{V(t+h,x(t+h))-V(t,x(t))\},$$

V satisfies:

$(i).\ V(t,0)=0,\quad \forall t\geq 0,$

$(ii).\ V(t,x)\geq W_1(x),\ \forall\ t\geq 0,\ \forall x\in U,$ for some continuous positive definite function W_1 on U

$(iii).\ \dot{V}\ (t,x)\leq 0,\ \ \forall t\geq t_0,\forall x\in U.$

Then the equilibrium $x=0$ is stable.

Proof

Given

Let $x=0$ be an equilibrium of the system $\dot{x}=f(t,x)$ and $U\subset D$ be a domain containing $x=0$.

To prove:

The equilibrium $x = 0$ is stable.

Fix an arbitrary $t_0 \geq 0$. Since V satisfies condition i and ii above and U is a domain containing 0, there exist a constant $r > 0$ and a strictly increasing continuous function α satisfying $\alpha(0) = 0$ such that $B_r \subset U$ and

$$\alpha(|x|) \leq V(t,x), \quad \forall t \geq t_0, \forall x \in B_r. \quad \ldots..(3.1.1)$$

Now let $\in > 0$ be arbitrary and fixed. In order to prove stability of $x = 0$ we need to show that there exist a $\delta = \delta(\epsilon, t_o)$ such that

$$|x(t_0) - X(t_0)| < \delta$$
$$|x(t) - X(t)| < \in, \quad \forall\, t \geq t_0 \geq 0$$

Holds. Begin by setting $\epsilon_1 = \min\{\in, r\}$ and choosing $\delta > 0$ such that

$$\sup_{|x|\leq\delta} V(t_0, x) =: \beta(t_0, \delta) < \alpha(\epsilon_1).$$

This can always be done because $\alpha(\epsilon_1) > 0$ and $\beta(t_0, \delta) \to 0$ as $\delta \to 0$.

Suppose we know that $|x(t_0)| < \delta$, and let τbe the smallest time t at which $|x(t)| \geq \epsilon_1$. This is well defined because x is a continuous function. Then by definition

$$|x(t)| \leq \epsilon_1, \;\; \forall t \in [t_0, t), \text{ and}$$
$$|x(\,t)| = \epsilon_1 \qquad (3.1.2)$$

Therefore, as $\epsilon_1 \leq r$ property (iii) gives that

$$\frac{d}{dt} V\big(t, x(t)\big) \leq 0, \quad \forall t \in [t_0, t),$$

Whence

$$V\big(\tau, x(\tau)\big) \leq V\big(t_0, x(t_0)\big) < \alpha(\,\epsilon_1),$$

But, by (3.1.1) and (3.1.2) we also have

$$V\big(\tau, x(\tau)\big) \geq \alpha(|x(\tau)|) = \alpha(\,\epsilon_1),$$

Giving a contradiction.

Therefore, no such τ can exist. Thus as $\epsilon_1 \leq \epsilon$, this shows that

$$|x(t_0)| < \delta$$
$$|x(t)| < \epsilon, \qquad \forall t \geq (t_0)$$

with $\epsilon > 0$ and $t_0 \geq 0$ arbitrary.

This proves the result.

3.2 Theorem

Let $x = 0$ be the solution of $\dot{x}(t) = f(t, x_t)$. Suppose that $f: \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ in $\dot{x}(t) = f(t, x_t)$ takes $\mathbb{R} \times$(bounded sets in $\mathcal{C}$) into bounded sets in $\mathbb{R}^n$,and $u, v, w: \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ are continuous non decreasing function with $u(s), v(s) > 0$ for all $s > 0, u(0) = v(0) = 0$ and v strictly increasing.

Suppose further that there exist a continuous function $V: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ such that

$$i.\;\; u(|x|) \leq V(t,x) \leq v(|x|), \;\; \forall t \in \mathbb{R}, \quad \forall x \in \mathbb{R}^n,$$
$$ii. \dot{V}\big(t, x(t)\big) \leq -w(|x(t)|)$$
$$if\; V(t+\theta, x(t+\theta)) \leq V\big(t, x(t)\big)$$

For $\theta \in [-r, 0]$ where $x(t)$ is any trajectory of $\dot{x}(t) = f(t, x_t)$. Then the solution $x = 0$ is uniformly stable.

Proof

Given

Let $x = 0$ be the solution of $\dot{x}(t) = f(t, x_t)$.

To prove:

The solution $x = 0$ is uniformly stable.

For arbitrary fixed $\epsilon > 0$ let us fix δ in the range $0 < \delta < v^{-1}\big(u(\epsilon)\big)$ which may be done by virtue of the assumed properties of the function u and v. Now take any function $\emptyset \in \mathcal{C}$ such that $\|\emptyset\| < \delta$.

Then according to property i,

We will have for any t_0

$$V(t_0 + \theta, \emptyset(\theta)) < v(|\emptyset(\theta)|) \leq v(\delta)$$

For all $\theta \in [-r, 0]$. Let x be the solution of $\dot{x}(t) = f(t, x_t)$ with initial data $x_{t0} = \emptyset$. What this then tells us is that

$$V\big(t_0 + \theta, x(t_0 + \theta)\big) \leq v(\delta) \text{ for all } \theta \in [-r, 0]$$

Suppose that τ is the earliest time after t_0 such that $V\big(\tau, x(\tau)\big) \leq v(\delta)$. As a consequence of the above, we now know that we must have

$$V\big(\tau + \theta, x(\tau + \theta)\big) < v(\delta) \text{ for all } \theta \in [-r, 0].$$

But then the condition within property ii of the theorem must be satisfied which means that we have $\dot{V}\big(\tau, x(\tau)\big) \leq 0$.

Therefore the continuity of $V\big(.,x(.)\big)$ implies that

$$V(t, x(t)) \leq v(\delta) < u(\epsilon)$$

For all $t \geq t_0 - r$. But condition i tells us that if $|x(t)| \geq \epsilon$

Then $V(t, x(t)) \geq u(\epsilon)$ and so we would have a contradiction.

Consequently, we conclude that in fact $|x(t)| < \epsilon$ for all $t \geq t_0 - r$.

Therefore this shows that given any $\epsilon > 0$ we can choose a $\delta > 0$ independently of t_0 such that whenever $\|x_{to}\| < \delta$ we must have $|x(t)| < \epsilon$ for all $t \geq t_0 - r$.

This is precisely the definition of uniform stability.

The theorem is proved.

CONCLUSION

In this dissertation **"Stability Theory Of Differential Equation To Power Control In Wireless Network".** we discussed about stability properties of solutions of linear second order differential equation with random coefficients. In particular we discussed about, Basic concepts of stability and instability, and some definitions. Stability properties, Stability properties of linear and nonlinear system and theorems. On small solutions of second order differential equation with random co-efficient theorems. Finally stability properties of solutions of linear second order differential equation with random coefficients.

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