**Original Research Article** **Open Access**

# Navir – Stokes Equation for two Dimentional in Fluid Dynamics

**Kiruthika R and Ramadevi S**

Department of Mathematics, Vivekanandha College of Arts & Sciences for women(Autonomous), Tiruchengode, Namakkal,Tamil Nadu, India.

## Abstract

We turn the Navier-stokes equations for a two dimensional viscous incompressible fluid into a system of functional integrals in the trajectory space of a suitable diffusion process. Using probabilistic techniques as Girsanov's transformation and Bismut-Elworthy formula. The large deviation principle is established for stochastic Navier-stokes equations with multiplicative noise for domains that can be unbounded. The large deviation principle is equivalent to the Laplace principle in our function space setting.

**Keywords** Navier-stokes equation**,** Stochastic Navier-stokes equation, Large deviation, Girsanov theorem**,** Stochastic differential equation

**INTRODUCTION**

We consider the Navier-stokes equations for velocity u and pressure p in a viscous, incompressible, planar fluid with initial velocity $u_0$ in the absence of external forces ,

$$\frac{\partial u}{\partial t} + (u.\nabla)u + \nabla p = v\Delta u \text{ for all (t,x)} \in \mathbb{R}^+ \times \mathbb{R}^2,$$

$$\nabla.u = 0 \text{ for all (t,x)} \in \mathbb{R}^+ \times \mathbb{R}^2,$$

With initial and boundary conditions

u(x,0) = $u_0$(x) for all x $\in \mathbb{R}^2$,

$\lim_{|x|\to\infty} u(t,x) = 0$ for all t $\in \mathbb{R}^+$

We recall the vorticity $\xi$ = rot u = $\partial_1 u^2 - \partial_2 u^1$ and the velocity

$$\frac{\partial \xi}{\partial t} + (\nabla^{\perp}\psi.\nabla)\xi = v\Delta\xi, \text{ (t,x)} \in \mathbb{R}^+ \times \mathbb{R}^2 \to (1)$$

The aim of this work is to analyze the above system with probabilistic methods. This was suggested to the author by Mark Freidlin. Using Probabilistic representations we transform above system in the problem

$\xi(t.x) = E[\xi_0(X_t^{t,x})]$ ,(t,x)$\in \mathbb{R}^+ \times \mathbb{R}^2$, → (2) where $dX_s^{t,x}$ = - u (t-s , $X_s^{t,x}$ ) ds+$\sqrt{2v}$d$w_s$, $X_0^{t,x}$ = x

Ben-Artzi in [1] showed that , for initial vorticity $\xi_0$ in $L^1$ ,the equation (1) has a long time solution. He prosved that this solution is unique and regular.

Initially, our plan was to recover the same results for system (2) by probabilistic techniques. We realize it just partially: we prove existence and uniqueness under the assumption that the initial vorticity belongs to $L^p \cap L^q$ with 1≤p < 2 < q,and we do not prove regularity results for the solution.

The theory of large deviations is an active and important topic in probability theory and has rightly received considerable attention. The framework for the theory along with important applications can be found in the book by varadhan [8].Wentzell-Freidlin type large deviation results for the two dimensional stochastic Navier-stokes equations with additive noise were proved by chang [7] using the Girsanov transformation In the present work ,the Wentzell-Freidlin large deviation principle is established for stochastic Navier-stokes equations with multiplicative noise for domains that can be unbounded. The methods employed in this paper are different from those in [7] and will extend to the stochastic magnato-hydrodynamic system introduced.

*Address Correspondence at Department of Mathematics, Vivekanandha College of Arts & Sciences for women (Autonomous), Tiruchengode, Namakkal, Tamil Nadu, India. Ph:+9659306496 Email:kiruthika9659@gmail.com

The work is organized as follows: In section 1,we turn the Navier-stokes system into the system of functional integrals (2). The second and third sections contain some preliminary results which we need in our proofs of the existence and uniqueness of the solution.

In section 2, we provide some useful estimates for the $L^p(\Omega)$ norms of the Girsanov densities corresponding to the stochastic differential equations with additive noise.

In section 3 and 4, the stochastic Navier-stokes equation has been studied by several authors (for example, Capinsky and Gatarek, Flandoli and Gatarek) in recent years.The aim of this sections is to establish large deviation principle for the family $\{u^{\varepsilon}\}$.

**1.0 Transformation of NS equations into functional integrals**

In this chapter, we discussed about the Probabilistic representation of the solution of poisson's problem in $\mathbb{R}^2$ and Probabilistic representations of the derivatives of the solution of poisson's problem in $\mathbb{R}^2$.

**1.1 Probabilistic representation of the solution**

**of poisson's problem in $\mathbb{R}^2$**

**1.1.1 Theorem**

Let f = $\mathbb{R}^2 \to \mathbb{R}^+$ be a continuous function , f ≠ 0 . Then for every x $\in \mathbb{R}^2$ ,we have $\int_0^{\infty} p_t(f)(x)\,dt = \infty$.

**Proof :**

There exists a > 0 such that $\{f > a\} \neq \phi$ .Since f is continuous function. There exist y in $\mathbb{R}^2$ and $\rho > 0$ such that B ( y , $\rho$ ) ⊂ $\{f > a\}$

Consider $x \in \mathbb{R}^2$

There exist $\beta, \alpha \in [0, 2\pi)$ and r , R ∈ ℝ such that $0 \le \alpha < \beta < 2\pi$ , $0 < r < R$ and $\{z/r < |z| < R, \alpha < \arg(z) < \beta\} \subset B(y - x, \rho)$ .

It follows that

$$P(W_t + x \in B(y, \rho)) = P(W_t \in B(y - x, \rho))$$

$$\ge \frac{(\beta - \alpha)}{2\pi}\left(\exp\left(-\frac{r^2}{2t}\right) - \exp\left(-\frac{R^2}{2t}\right)\right)$$

And $\int_0^\infty E[f(W_t + x)]\,dt \ge$ a $\int_0^\infty \frac{(\beta-\alpha)}{2\pi}\left(\exp\left(-\frac{r^2}{2t}\right) - \exp\left(-\frac{R^2}{2t}\right)\right) dt$

Since $\frac{(\beta-\alpha)}{2\pi}\left(\exp\left(-\frac{r^2}{2t}\right) - \exp\left(-\frac{R^2}{2t}\right)\right) \sim \frac{1}{t}$

$$\int_0^\infty p_t(f)(x)\,dt = \infty.$$

### 1.2 Probabilistic representations of the derivatives of the solution of poisson's problem in $\mathbb{R}^2$

### 1.2.1Results

(i) For all p ∈ [ 1 , ∞) there exists a constant $c_p$ such that

$$E[\,|W_t^{(i)}|^p\,]^{\frac{1}{p}} \le c_p\sqrt{t} \text{ for all } t \in [0, \infty).$$

(ii) For all r ∈ ( 1 , ∞) there exists a constant c such that

$E[\,|f(x + W_t)W_t^{(i)}|\,] \le c_r\,\|f\|_r\, t^{-\frac{1}{r}+\frac{1}{2}}$ for all $f \in L^r$ , for all t > 0.

### 1.2.2 Theorem

Let $f \in L^p \cap L^q$ with $1 \le p < 2 < q \le \infty$. Then the integrals

$\psi_i(x) = \int_0^\infty \frac{1}{t} E[f(x + W_t)\,W_t^{(i)}]\,dt$, i = 1 , 2 Converge and there exist a constant $c_{p,q}$ such that $|\psi_i(x)| \le c_{p,q}\,\|f\|_{p,q}$ for all $x \in \mathbb{R}^2$.

**Proof :**

It is enough to prove the theorem for $1 < p < 2 < q < \infty$. By the preceding result, for all t > 0 , we have

$$\frac{1}{t} E[\,|f(x + W_t)\,W_t^{(i)}|\,] \le c_{p'}\,\|f\|_p\, t^{-\frac{1}{p}-\frac{1}{2}} I_{(1,\infty)}(t) + c_{q'}\,\|f\|_q\, t^{-\frac{1}{q}-\frac{1}{2}} I_{(0,1]}(t)$$

Since $-\frac{1}{p} - \frac{1}{2} < -1$ and $-\frac{1}{q} - \frac{1}{2} > -1$ , the function on the right hand side is integrable.Therefore the integrals $\psi_i(x)$ converge and

$$|\psi_i(x)| \le c_{p'}\,\|f\|_p \int_1^\infty t^{-\frac{1}{p}-\frac{1}{2}}\,dt + c_{q'}\,\|f\|_q \int_0^1 t^{-\frac{1}{q}-\frac{1}{2}}\,dt$$

$$= c_{p'}\|f\|_p\left(\frac{1}{p} + \frac{1}{2} - 1\right) + c_{q'}\|f\|_q\left(1 - \frac{1}{q} - \frac{1}{2}\right)$$

$$= c_{p'}\,\|f\|_p\left(\frac{1}{p} - \frac{1}{2}\right) + c_{q'}\,\|f\|_q\left(\frac{1}{2} - \frac{1}{q}\right)$$

### 1.2.3 Theorem

If $f \in L^p \cap L^q$ with $1 < p < 2 < q$ then the functions $\psi_i(x)$ are uniformly continuous .

**Proof :**

Consider the inequality ,

$$\int_0^\infty \left|\frac{1}{t} E\left[f(x + W_t)\,W_t^{(i)}\right] - \frac{1}{t} E\left[f(\bar{x} + W_t)\,W_t^{(i)}\right]\right| dt$$

$$\le c_1\,\|\tau_{x-\bar{x}} f - f\|_p + c_2\,\|\tau_{x-\bar{x}} f - f\|_q$$

With the uniform continuity of the shift in $L^r$ for r ∈ (1, ∞). Therefore the functions $\psi_i(x)$ are uniformly continuous .

### 1.2.4 Theorem

If $f \in L^p \cap L^q$ with $1 \le p < 2 < q$ then the functions $\psi_i(x)$ are in $c_0$ .

**Proof :**

For all $R > 0$ , we have,

$$\frac{1}{t} E[\,|f(x + W_t)\,W_t^{(i)}|\,]$$

$$= \frac{1}{t} E[\,|f(x + W_t)\,W_t^{(i)}|\, I_{\{|W_t| \le R\}}] + \frac{1}{t} E[\,|f(x + W_t)\,W_t^{(i)}|\, I_{\{|W_t| > R\}}]$$

Consider the term ,

$$\frac{1}{t} E[\,|f(x + W_t)\,W_t^{(i)}|\, I_{\{|W_t| \le R\}}]$$

$$\le c_p\,\|f\, I_{\{|y-x| \le R\}}\|_p\, t^{-\frac{1}{p}-\frac{1}{2}} I_{(1,\infty)}(t) + c_q\,\|f\, I_{\{|y-x| \le R\}}\|_q\, t^{-\frac{1}{q}-\frac{1}{2}} I_{(0,1)}(t)$$

Hence , for all fixed R ,

$$\int_0^\infty \frac{1}{t} E\left[\,|f(x + W_t)\,W_t^{(i)}|\, I_{\{|W_t| \le R\}}\right] dt$$

$$\le c_1\|f\, I_{\{|y-x| \le R\}}\|_p + c_2\,\|f\, I_{\{|y-x| \le R\}}\|_q$$

The above inequality tends to 0 as $|x| \to \infty$.

For all $|x| > R$ ,the inclusion $\{|y - x| \le R\} \subset \{|y| > |x| - R\}$ holds .

$$\int_0^\infty \frac{1}{t} E\left[\,|f(x + W_t)\,W_t^{(i)}|\, I_{\{|W_t| \le R\}}\right] dt \to 0$$

Next consider the term ,

$$\frac{1}{t} E[\,|f(x + W_t)\,W_t^{(i)}|\, I_{\{|W_t| > R\}}]$$

For all t > 0 we get ,

$$\sup_x\left(\frac{1}{t} E[\,|f(x + W_t)\,W_t^{(i)}|\, I_{\{|W_t| > R\}}]\right) \le \frac{1}{t}\, t^{\frac{-1}{q}}\,\|f\|_q\, t^{\frac{1}{2}} I_{(0,1]} + \frac{1}{t}\, t^{\frac{-1}{p}}\,\|f\|_p\, t^{\frac{1}{2}} I_{(1,\infty)}$$

Since the function of t on the left hand side is dominated and converges to 0 as R → ∞ . By the dominated convergence theorem

, $\sup_x\left(\int_0^\infty \frac{1}{t} E\left[\,|f(x + W_t)\,W_t^{(i)}|\, I_{\{|W_t| > R\}}\right] dt\right) \to 0$ as R → ∞

Fix an arbitrary $\epsilon > 0$ .Choosing R so that for each x ,

$$\int_0^\infty \frac{1}{t} E\left[\,|f(x + W_t)\,W_t^{(i)}|\, I_{\{|W_t| > R\}}\right] dt \le \epsilon$$

We obtain ,

$$\lim_{|x|\to\infty} \int_0^\infty \frac{1}{t} E[\,|f(x + W_t)\,W_t^{(i)}|\,]\,dt$$

$$\le \lim_{|x|\to\infty} \int_0^\infty \frac{1}{t} E[\,|f(x + W_t)\,W_t^{(i)}| I_{\{|W_t| \le R\}}]\,dt + \lim_{|x|\to\infty} \int_0^\infty \frac{1}{t} E[\,|f(x + W_t)\,W_t^{(i)}| I_{\{|W_t| \le R\}}]$$

$$\lim_{|x|\to\infty} \int_0^\infty \frac{1}{t} E[\,|f(x + W_t)\,W_t^{(i)}|\,]\,dt \le \epsilon$$

## 2.0 The girsanov densities

### 2.1 Definition

A real-valued stochastic process {B(t) : t ≥ 0} is called a Brownian motion with start in x ∈ ℝ if the following holds :

(i) B(0) = x .

(ii) The process has independent increments (i.e) for all times $0 \le t_1 \le t_2 \le \ldots\ldots \le t_n$ the increments $B(t_n) - B(t_{n-1})$ , $B(t_{n-1}) - B(t_{n-2})$ , . . . . . . . . . , $B(t_2) - B(t_1)$ are independent random variables .

(iii) For all t ≥ 0 and h > 0 , the increments B(t + h) – B(t) are

are normally distributed with expectation zero and variance h .

(iv) Almost surely , the function t → $B(t)$ is continuous .

**2.2 Definition**

We say that the process X = (X(t) : t ≥ 0) is adapted to the process Z = (Z(t) : t ≥ 0) if , for each t ≥ 0 , there exist a function $g_t(.)$ such that X(t) = $g_t$ (Z(s) : $0 \le s \le t$) .

**2.3 Definition**

A real-valued process $X_t, t \in T$ , adapted to $(\mathcal{T}_t)$ is a submartingale (with respect to $\mathcal{T}_t$ ) if

(i) E $[X_t^+] < \infty$ for every t ∈ $T$ .

(ii) E $[X_t \mid \mathcal{T}_s] \geq X_s$ $a.s$ for every pair s,t such that s < t .

(iii) A process X such that −X is a submartingale is called a supermartingale and a process which is both a submartingale and a supermartingale is a martingale .

**2.4 Definition**

X is said to be semimartingale if $X_t = M_t + A_t$ where $M_t$ is a continuous local martingale and $A_t$ is a continuous adapted process that is locally of bounded variation .

**2.5 Definition**

A stochastic process $(M_t)_{t\geq0}$ is called a local martingale (with respect to the filtration $(\mathcal{F}_t)_{t\geq0}$) ) if there is a sequence of stopping times $(T_n)_{n\geq0}$ such that

(i) The sequence $(T_n)_{n\geq0}$ is increasing and almost surely satisfies $\lim_{n\to\infty} T_n = \infty$ .

(ii) For n ≥ 1 , the process $(M_{t\wedge T_n})_{t\geq0}$ is a uniformly integrable martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq0}$ .

**2.6 Theorem**

If Q ⊲ P and if D is continuous , every continuous ($\mathcal{F}_t^0$ , P)- semimartingale is acontinuous ($\mathcal{F}_t^0$, Q)-semimartingale .More precisely, if M is a continuous ($\mathcal{F}_t^0$, P)-local martingale then $\widetilde{M}$=

M $-D^{-1}\langle M, D\rangle$ is a continuous ($\mathcal{F}_t^0$ , Q)-local martingale .Moreover , if N is another continuous P-local martingale ,$\langle \widetilde{M}, \widetilde{N}\rangle = \langle \widetilde{M}, N\rangle$ = ⟨M,N⟩. It's the Girsanov theorem.

**Proof :**

If X is a cadlag process and if XD is a ($\mathcal{F}_t^0$, P)-local martingale .Then X is a ($\mathcal{F}_t^0$, Q)-local martingale

$D_t^T$ is a density of Q with respect to P on ($\mathcal{F}_{T\wedge r}^0$) and if $(XD)^T$ is a p-martingale , it is easily seen that $X^T$ is a Q-martingale . A sequence of ($\mathcal{F}_t^0$)stopping times increasing to $\infty$ P-almost surely increases also to $\infty$ Q-almost surely .

Let $T_n$ = inf {t: $D_t \le \frac{1}{n}$}.It is easy to see that the process $(D^{-1}.\langle M, D\rangle)^{T_n}$ is p-almost surely finite and consequently $(\widetilde{M}D)^{T_n}$ is the product of two semimartingales

It follows that ,

$$(\widetilde{M}D)_t^{T_n} = M_0D_0 + \int_0^{T_n\wedge t} \widetilde{M}_s dD_s + \int_0^{T_n\wedge t} D_s d\widetilde{M}_s + \langle \widetilde{M}, D\rangle_{T_n\wedge t}$$

$$= M_0D_0 + \int_0^{T_n\wedge t} \widetilde{M}_s dD_s + \int_0^{T_n\wedge t} D_s dM_s - \langle M, D\rangle_{T_n\wedge t} + \langle \widetilde{M}, D\rangle_{T_n\wedge t}$$

$$= M_0D_0 + \int_0^{T_n\wedge t} \widetilde{M}_s dD_s + \int_0^{T_n\wedge t} D_s dM_s$$

Which proves that $(\widetilde{M}D)^{T_n}$ is a P-local martingale and it also a Q-local martingale .

But $(T_n)$ increases Q-almost surely to $\infty$ ,and a process which is locally a local martingale .

$\widetilde{M}$ = M $-D^{-1}\langle M, D\rangle$ is a continuous ($\mathcal{F}_t^0$, Q)-local martingale

And follows from the fact that the bracket of a process of finite variation with any semimartingale vanishes identically .

$$\langle \widetilde{M}, \widetilde{N}\rangle = \langle \widetilde{M}, N\rangle = \langle M,N\rangle$$

**2.7 Definition**

A continuous time process $X_t$ is said to be a continuous local martingale with respect to $\{\mathcal{F}_t , t \geq 0\}$ if there are stopping times $T_n \uparrow \infty$ such that $X_t^{T_n} = X_{T_n\wedge t}$ on $\{T_n > 0\}$ on $\{T_n > 0\}$

$X_t^{T_n}$ = 0 on $\{T_n = 0\}$

is a martingale with respect to $\{\mathcal{F}_{t\wedge T_n} : t \geq 0\}$ . The stopping times $\{T_n\}$ are said to reduce X .

**2.8 Theorem**

For p ≥ 2 , there exist a constant $c_p$ such that for any continuous local martingale M such that $M_0 = 0$ , E$[(M_\infty^*)^p]$ ≤ $c_p$E$[\langle M, M\rangle_\infty^{p/2}]$

**Proof :**

It is enough to prove the result for bounded M .

The function x → $|x|^p$ being twice differentiable and

$$|M_\infty|^p = \int_0^\infty p|M_s|^{p-1} (sgn\ M_s) dM_s + \frac{1}{2}\int_0^\infty p(p-1)|M_s|^{p-2} d\langle M, M\rangle_s$$

Consequently,

$$E[|M_\infty|^p] = \frac{p(p-1)}{2} E\left[\int_0^\infty |M_s|^{p-2} d\langle M, M\rangle_s\right]$$

$$\le \frac{p(p-1)}{2} E\left[(M_\infty^*)^{p-2}\langle M, M\rangle_\infty\right]$$

$$\le \frac{p(p-1)}{2} \|(M_\infty^*)^{p-2}\|_{p/p-2}\|\langle M, M\rangle_\infty\|_{p/2}$$

On the other hand ,

By the doop's inequality ,

$$\|M_\infty^*\|_p \le \left(\frac{p}{p-1}\right) \|M_\infty\|_p$$

and the result follows from straightforward

$$E\left[(M_\infty^*)^p\right] \le c_p\ E\left[\langle M, M\rangle_\infty^{p/2}\right]$$

**2.9 Theorem**

For p ≥ 2 , there exist a constant $c_p$ such that $c_p$ E $[\langle M, M\rangle_\infty^{p/2}] \le$ E $[(M_\infty^*)^p]$

**Proof :**

It is enough to prove the result in the case when ⟨ M , M ⟩ is bounded

Consider $a_p$ will always designate a universal constant. But this constant may vary from line to line . For instance , for two reals x and y

$$|x + y|^p \le a_p(|x|^p + |y|^p)$$

From the equality , $M_t^2 = 2\int_0^t M_s dM_s + \langle M, M\rangle_t$

It follows that ,

$$E\left[\langle M, M\rangle_\infty^{p/2}\right] \le a_p\left( E\left[(M_\infty^*)^p\right] + E\left[\left|\int_0^\infty M_s dM_s\right|^{p/2}\right]\right)$$

and applying the theorem (2.8) to the local martingale $\int_0^t M_s dM_s$ we get

$$E\left[\langle M, M\rangle_\infty^{p/2}\right] \le a_p\left( E\left[(M_\infty^*)^p\right] + E\left[\left(\int_0^\infty M_s^2\, d\langle M, M\rangle_s\right)^{p/4}\right]\right)$$

$\leq \ a_p ( \mathrm{E}\,[(M_\infty^*)^p] + ( \mathrm{E}[(M_\infty^*)^p]\mathrm{E}[\langle M,M\rangle_\infty^{p/2}] )^{1/2})$

If we consider x = $\mathrm{E}\,[\langle M,M\rangle_\infty^{p/2}]^{1/2}$ and y = $\mathrm{E}\,[(M_\infty^*)^p]^{1/2}$. Therefore , the above inequality we get

$$x^2 - a_p xy - a_p y^2 \leq 0$$

Which entails that x is less than or equal to the positive root of the equation $x^2 - a_p xy - a_p y^2 = 0$. Which is of the form $a_p y$

$$c_p \ \mathrm{E}\,[\langle M,M\rangle_\infty^{p/2}] \leq \mathrm{E}\,[(M_\infty^*)^p]$$

**2.10 Theorem**

Let p ∈ (0, ∞) . there exist two positive constants $c_p$ and $C_p$ such that , for all continuous local martingales M vanishing at zero , E[ $\langle M,M\rangle_\infty^{p/2}$ ] ≤ $c_p$ E[ $M_\infty^*$ ] ≤ $C_p$ E[ $\langle M,M\rangle_\infty^{p/2}$ ].Which is the Burkholder-Davis-Gundy inequality.

[Note that the theorems (2.8),(2.9)]

**3.0 Stochastic Navier-stokes equations**

**3.1 Definition**

A stochastic process {W(t) : $0 \leq t \leq T$} is said to be an H-valued { $\mathcal{F}_t$ }-adapted Wiener process with covariance operator Q if

(i) for each non-zero h ∈ $H$ , $|Q^{1/2}h|^{-1}$ (W(t) , h) is a standard one dimensional Wiener process and

(ii) for any h ∈ $H$ , (W(t) , h) is a martingale adapted to $\{\mathcal{F}_t\}$ .

**3.2 Theorem**

For any real-valued functions $\phi$ and $\psi$ with compact support in $\mathcal{R}^2$, the following hold :

1 . $|\phi^2\psi^2|_{L^1} \leq \ |\phi|_{L^2}|\psi\,|_{L^2}|\nabla\phi|_{L^2}|\nabla\psi\,|_{L^2}$

2 . $|\phi|_{L^4}^4 \leq \frac{1}{2}|\phi|_{L^2}^2|\nabla\phi|_{L^2}^2$

**Proof :**

For any (x , y) , the function can be written as

$$\phi(x,y) = \int_{-\infty}^{x} \partial_1\phi(s\,,y)ds$$

$$= -\int_{x}^{\infty} \partial_1\phi(s\,,y)ds$$

$$\psi\ (x\ ,\ y) = \int_{-\infty}^{y} \partial_2\psi(x\,,t)dt$$

$$= -\int_{y}^{\infty} \partial_2\psi(x\,,t)dt$$

If we consider x = $\mathrm{E}\,[\langle M,M\rangle_\infty^{p/2}]^{1/2}$ and y = $\mathrm{E}\,[(M_\infty^*)^p]^{1/2}$. Therefore , the above inequality we get

$$x^2 - a_p xy - a_p y^2 \leq 0$$

Which entails that x is less than or equal to the positive root of the equation $x^2 - a_p xy - a_p y^2 = 0$. Which is of the form $a_p y$

$$c_p \ \mathrm{E}\,[\langle M,M\rangle_\infty^{p/2}] \leq \mathrm{E}\,[(M_\infty^*)^p]$$

**2.10 Theorem**

Let p ∈ (0, ∞) . there exist two positive constants $c_p$ and $C_p$ such that , for all continuous local martingales M vanishing at zero , E[ $\langle M,M\rangle_\infty^{p/2}$ ] ≤ $c_p$ E[ $M_\infty^*$ ] ≤ $C_p$ E[ $\langle M,M\rangle_\infty^{p/2}$ ].Which is the Burkholder-Davis-Gundy inequality.

[Note that the theorems (2.8),(2.9)]

**2.10.1 Lemma**

For each q ∈ ( 2 , ∞ ) , there exist a constant c > 0 such tha for all u ∈ $BC$ ( [ 0 , ∞ ) , $c_p$ ) ,

E [ $|z_t^{x,u}|^{\,q}$ ] ≤ c exp( c ( $\frac{\|u\|_\infty}{\mu}$ )$^q$ $t^{\frac{q}{2}}$ ) for all t ∈ [ 0 , ∞ ) , for all x ∈ $\mathbb{R}^2$ .

**Proof :**

Since $dz_s^x = \frac{1}{\mu}$ u ( s , x + $\mu w_s$) $z_s^x$ $dw_s$

E [ $|z_t^{x,u}|^{\,q}$ ] ≤ $c_q$ ( 1 + E [ $\left|\int_0^t \frac{1}{\mu}\,\mathrm{u}\,(s\,,x + \mu w_s)\, z_s^{x,u}\, dw_s\right|^q$ ] ) for all t ≥ 0 by the theorem (2.10) we get ,

E[ $|z_t^{x,u}|^{\,q}$ ] ≤ $c_q$ ( 1 + c $\frac{1}{u^q}$ E [( $\int_0^t |\mathrm{u}\,(s\,,x + \mu w_s)|^2 |z_s^{x,u}\,|^2\, ds\,)^{\frac{q}{2}}$ ] ) for all t ≥ 0

and from Holder 's inequality

E[ $|z_t^{x,u}|^q$ ] ≤ $c_q$ $+c_q^{'}$ $\frac{1}{\mu^q}$ E[ $t^{(q/2)-1}$ $\int_0^t |\mathrm{u}(s\,,x + \mu w_s)|^q |z_s^{x,u}\,|^q\, ds$ ]

$\leq c_q + c_q^{'}\ (\frac{\|u\|_\infty}{\mu})^q\, t^{(\frac{q}{2})-1} \int_0^t E\,[\,|z_s^{x,u}\,|^q\,]$ ds

Applying Gronwall ' s lemma we get ,

E [ $|z_t^{x,u}|^{\,q}$ ] ≤ $c_q$ exp ( $c_q^{'}$ ( $\frac{\|u\|_\infty}{\mu}$ )$^q$ $t^{\frac{q}{2}}$ ) for all t ≥ 0

**2.10.2 Lemma**

For each q ∈ ( 2 , ∞ ) , there exist a constant c such that for all u , $v \in BC$ ( [ 0 , ∞ ) , $c_p$ ) ,

E [ $|z_t^{x,u} - z_t^{x,v}\,|^{\,q}$ ] ≤ c ( $\frac{\|u-v\|_\infty}{\mu}$ )$^q$ $t^{\frac{q}{2}}$ ) exp( c $\frac{\|u\|_\infty^q + \|v\|_\infty^q}{\mu^q}$ $t^{\frac{q}{2}}$ )

for all t ∈ [ 0 , ∞ ) , for all x ∈ $\mathbb{R}^2$.

**Proof :**

since $z_t^{x,u}$ solves ,

$dz_s^x = \frac{1}{\mu}$ u ( s , x + $\mu w_s$) $z_s^x$ $dw_s$ , $z_0^x$ = 1

and $z_t^{x,v}$ solves ,

$dz_s^x = \frac{1}{\mu}$ v ( s , x + $\mu w_s$) $z_s^x$ $dw_s$ , $z_0^x$ = 1 ,we have

E [ $|z_t^{x,u} - z_t^{x,v}\,|^{\,q}$ ] ≤ $c_q$ E [ $\left|\int_0^t \frac{1}{\mu}(\mathrm{u} - \mathrm{v})(s\,,x + \mu w_s)\, z_s^{x,u}\, dw_s\right|^q$ ] $+c_q$ E [ $\left|\int_0^t \frac{1}{\mu}\,\mathrm{v}\,(s\,,x + \mu w_s)\,(\,z_s^{x,u} - z_s^{x,v}\,)\ dw_s\right|^q$]

Applying the theorem(2.10) we get ,

E[$|z_t^{x,u} - z_t^{x,v}\,|^{\,q}$]≤
$c_q^{'}\ \frac{1}{\mu^q}$E[( $\int_0^t |(\mathrm{u}-\mathrm{v})(s\,,x + \ \mu w_s)|^2\ |z_s^{x,u}|^2$ ds )$^{q/2}$]
$+\, c_q^{'}\ \frac{1}{\mu^q}$ E [( $\int_0^t |\,\mathrm{v}\,(s\,,x + \ \mu w_s)|^2\ |z_s^{x,u} - \ z_s^{x,v}\,|^2$ dr )$^{q/2}$]

By the Holder 's inequality

E [ $|z_t^{x,u} - z_t^{x,v}\,|^{\,q}$ ] ≤

$c_q^{'}\ \frac{1}{\mu^q}$ E [ $t^{(q/2)-1} \int_0^t |(\mathrm{u}-\mathrm{v})\,(s\,,x + \mu w_s)|^q\,|\,z_s^{x,u}\,|^q\, ds$ ] + $c_q^{'}\ \frac{1}{\mu^q}$ E [ $t^{(q/2)-1} \int_0^t |\,\mathrm{v}\ (s\,,x + \mu w_s)|^q\,|\,z_s^{x,u} - z_s^{x,v}\,|^q\, dr$ ]

$\leq c_q^{'}\ \frac{1}{\mu^q}\ t^{(q/2)-1}\,\|u-v\|_\infty^q \int_0^t E\,[\ |\,z_s^{x,u}\,|^q\,]$ ds

$+c_q^{'}\ \frac{1}{\mu^q}\ t^{(q/2)-1}\,\|v\|_\infty^q \int_0^t E\,[\ |\,z_s^{x,u} - \ z_s^{x,v}\,|^q\,]$ ds

$\leq\ c_q^{'}\ \frac{1}{\mu^q}\ t^{(q/2)-1}\,\|\,u - v\,\|_\infty^q$ t c exp

( c ( $\frac{\|\,u\,\|_\infty}{\mu}$ )$^q$ $t^{q/2}$ ) + $c_q^{'}\ \frac{1}{\mu^q}\ t^{(q/2)-1}\,\|\,v\,\|_\infty^q$
$\int_0^t E\,[\ |\,z_s^{x,u} - \ z_s^{x,v}\,|^q\,]$ ds

Applying Gronwall ' s lemma ,we obtain

E[ $|z_t^{x,u} - z_t^{x,v}|^q$ ] $\leq$ c ( $\frac{\| u - v \|_\infty}{\mu}$ $)^q$ $t^{\frac{q}{2}}$ ) exp ( c $\frac{\| u \|_\infty^q + \| v \|_\infty^q}{\mu^q}$ $t^{\frac{q}{2}}$ ) for all t ∈ [ 0 , ∞ ) , for all x ∈ $\mathbb{R}^2$ .

## 3.0 Stochastic Navier-stokes equations

### 3.1 Definition

A stochastic process {W(t) : 0 $\leq t \leq T$} is said to be an H-valued $\{\mathcal{F}_t\}$-adapted Wiener process with covariance operator Q if

(i) for each non-zero h ∈ $H$ , $|Q^{1/2}h|^{-1}$ (W(t) , h) is a standard one dimensional Wiener process and

(ii) for any h ∈ $H$ , (W(t) , h) is a martingale adapted to $\{\mathcal{F}_t\}$ .

### 3.2 Theorem

For any real-valued functions $\phi$ and $\psi$ with compact support in $\mathcal{R}^2$, the following hold :

1 . $|\phi^2\psi^2|_{L^1} \leq |\phi|_{L^2}|\psi|_{L^2}|\nabla\phi|_{L^2}|\nabla\psi|_{L^2}$

2 . $|\phi|_{L^4}^4 \leq \frac{1}{2}|\phi|_{L^2}^2|\nabla\phi|_{L^2}^2$

**Proof :**

For any (x , y) , the function can be written as

$$\phi(x,y) = \int_{-\infty}^{x} \partial_1\phi(s,y)ds$$
$$= -\int_{x}^{\infty} \partial_1\phi(s,y)ds$$
$$\psi(x,y) = \int_{-\infty}^{y} \partial_2\psi(x,t)dt$$
$$= -\int_{y}^{\infty} \partial_2\psi(x,t)dt$$

by using the first equation above we have ,

$$|\phi(x,y)| = \frac{1}{2}\left|[\int_{-\infty}^{x}\partial_1\phi(s,y)ds + \int_{x}^{\infty} -\partial_1\phi(s,y)ds]\right|$$
$$\leq \frac{1}{2}\int_{-\infty}^{\infty}|\partial_1\phi(s,y)|ds \quad \rightarrow (1)$$
$$|\psi(x,y)| = \frac{1}{2}\left|[\int_{-\infty}^{y}\partial_2\psi(x,t)dt + \int_{y}^{\infty} -\partial_2\psi(x,t)dt]\right|$$
$$\leq \frac{1}{2}\int_{-\infty}^{\infty}|\partial_2\psi(x,t)|dt \quad \rightarrow (2)$$

Using (1) and (2) , and integrating with respect to x and y we get

$$|\phi\psi|_{L^1} \leq \frac{1}{4}|\partial_1\phi|_{L^1}|\partial_2\psi|_{L^1} \quad \rightarrow (3)$$

In the above , using $\phi^2$ in place of $\phi$ , and $\psi^2$ in place of $\psi$ , we have

$$|\phi^2\psi^2|_{L^1} \leq |\phi\partial_1\phi|_{L^1}|\psi\partial_2\psi|_{L^1} \quad \rightarrow (4)$$

Using the Cauchy – Schwarz inequality twice

$$|\phi^2\psi^2|_{L^1} \leq |\phi|_{L^2}|\psi|_{L^2}|\partial_1\phi|_{L^2}|\partial_2\psi|_{L^2} \quad \rightarrow (5)$$
$$|\phi^2\psi^2|_{L^1} \leq |\phi|_{L^2}|\psi|_{L^2}|\nabla\phi|_{L^2}|\nabla\psi|_{L^2} \quad \rightarrow (6)$$

Next to prove , $|\phi|_{L^4}^4 \leq \frac{1}{2}|\phi|_{L^2}^2|\nabla\phi|_{L^2}^2$

Put $\phi = \psi$ in the equation (5) and using Young's inequality

$$|\phi|_{L^4}^4 \leq \frac{1}{2}|\phi|_{L^2}^2|\nabla\phi|_{L^2}^2$$

### 3.3 Theorem

For a given r > 0 , let $B_r$ denote the $L^4$(G) ball in V : $B_r$ = {v ∈ $V$: $\|V\|_{L^4(G)} \leq r$}.Define the nonlinear operator F on V by F(u) = $-vAu - B(u)$ .Then for any $0 < \varepsilon < \frac{v}{2L}$ where L is the constant that appears in the condition . For all t ∈ $(0,T)$ , there exist a positive constant L such that for all u , v ∈ $V$, $|\sigma(t,u) - \sigma(t,v)|_{L_Q}^2 \leq L(\|u - v\|^2)$ , the pair (F, $\sqrt{\varepsilon}\sigma$ ) is monotone in $B_r$ .That is for any u ∈ V and v ∈ $B_r$ ,if w denotes u – v , then ⟨F(u) – F(v),w⟩ $-\frac{r^4}{v^3}$ $|w|^2 + \varepsilon|\sigma(t,u) - \sigma(t,v)|_{L_Q}^2 \leq 0$

**Proof :**

First , it is clear that ⟨Aw , w⟩ = $\|w\|^2$ .

Using b(u,v,w) = − b(u,w,v) and the bilinearity of the operator B ,

It follows that ,

$$\langle B(u), w\rangle = -\langle B(u,w), v\rangle$$
$$\langle B(v), w\rangle = -\langle B(v,w), v\rangle$$

Using the above two equations we have ,

$$\langle B(u) - B(v), w\rangle = -\langle B(w), v\rangle$$

Using the Holder's inequality ⟨B(u)−B(v) , w⟩ $\leq \|w\|_{L^4(G)}\|w\|\|V\|_{L^4(G)}$

Since , $|u|_{L^4}^4 \leq |u|^2\|u\|^2$

$$\langle B(u) - B(v), w\rangle \leq |w|^{1/2}\|w\|^{3/2}\|V\|_{L^4(G)}$$
$$\leq \frac{v}{2}\|w\|^2 + \frac{27}{32v^3}|w|^2\|V\|_{L^4(G)} \quad \rightarrow (7)$$

Where the last inequality follows from the fact that for any two real numbers a , b and any p , q > 1 with ,$\frac{1}{p} + \frac{1}{q} = 1$ , ab $\leq$,$\frac{|a|^p}{p} + \frac{|b|^q}{q}$

using the equation (7) and the operator F we get ,

$$\langle F(u) - F(v), w\rangle \leq -\frac{v}{2}\|w\|^2 + \frac{r^4}{v^3}|w|^2$$

Using the following condition ,

For all t ∈ $(0,T)$ , there exist a positive constant L such that for all u , v ∈ $V$, $|\sigma(t,u) - \sigma(t,v)|_{L_Q}^2 \leq L(\|u - v\|^2)$ and $\varepsilon < \frac{v}{2L}$ we get , ⟨F(u) − F(v),w⟩ $-\frac{r^4}{v^3}$ $|w|^2 + \varepsilon|\sigma(t,u) - \sigma(t,v)|_{L_Q}^2 \leq 0$

### 3.4 Theorem

Let f be in $L^2([0,T]:H)$ and let E($|u(0)|^2$) < ∞ . Let $u_n^\varepsilon$ denote the unique strong solution of the finite system of equation d($u_n^\varepsilon(t), v$) = {⟨f(t) , v⟩ + (F($u_n^\varepsilon(t)$),v)}dt +$\sqrt{\varepsilon}$ ($\sigma_n(t, u_n^\varepsilon(t))$ d$W_n(t), v$).With $u_n^\varepsilon(0)$ = $p_n u(0)$ . in C([0,T]:$H_n$) . Then , with K as in condition

For all t ∈ $(0,T)$ , there exist a positive constant K such that $|\sigma(t,u)|_{L_Q}^2 \leq K(1 + \|u\|^2)$ , the following conditions hold

1 . For all < $\frac{v}{2k} \wedge \frac{1}{2k^2}$ , and $0 \leq t \leq T$ ,

$$E(|u_n^\varepsilon(t)|^2) + v\int_0^t E\|u_n^\varepsilon(s)\|^2 ds \leq E|u(0)|^2 + \frac{2}{v}E\int_0^t|f(s)|_{v'}^2 ds + \varepsilon KT \quad \rightarrow (8)$$

and

$$E(\sup_{0\leq t\leq T}|u_n^\varepsilon(t)|^2 + v\int_0^T\|u_n^\varepsilon(s)\|^2 ds) \leq C(E|u(0)|^2, E\int_0^T|f(s)|_{v'}^2 ds, v, T) \quad \rightarrow (9)$$

2 . Let $\delta > 0$ and < $\frac{v}{2k}$ . Then

$$E(|u_n^\varepsilon(t)|^2)e^{-\delta t} + v\int_0^t E\|u_n^\varepsilon(s)\|^2 e^{-\delta s}ds$$
$$\leq E|u(0)|^2 + \frac{1}{\delta}E\int_0^t|f(s)|_{v'}^2 ds + \frac{\varepsilon K}{\delta}. \quad \rightarrow (10)$$

3 . If E$|u(0)|^4$ < ∞ , f is in $L^4([0,T]:V')$ , and $\varepsilon < \frac{v}{4k}$ ,then for all $0 \leq t \leq T$ , E($|u_n^\varepsilon(t)|^4$)$e^{-\delta t}$ +

$$3v\int_0^t E\|u_n^\varepsilon(s)\|^2\ |u_n^\varepsilon(s)|^2 e^{-\delta s}ds \leq$$

$$E|u(0)|^4 + C_\delta E\int_0^t|f(s)|_{v'}^4 e^{-\delta s}ds + \frac{4\varepsilon K}{\delta}\sup_{0\leq t\leq T}E(|u_n^\varepsilon(s)|^2\ ) \rightarrow (11)$$

**Proof :**

First to prove ,

$$E(\sup_{0\le t\le T}|u_n^\varepsilon(t)|^2 + v\int_0^T \|u_n^\varepsilon(s)\|^2\, ds)$$

$$\le C(E|u(0)|^2 , E\int_0^T |f(s)|_{v'}^2\, ds , v , T)$$

$$(d|u_n^\varepsilon(t)|^2 + 2v\|u_n^\varepsilon(t)\|^2)$$

$$= (2\langle f(t), u_n^\varepsilon(t)\rangle$$

$$+\varepsilon\, tr(\sigma_n(t,u_n^\varepsilon(t))Q\sigma_n(t,u_n^\varepsilon(t)))) dt$$

$$+ 2\sqrt{\varepsilon}\,(\sigma_n(t,u_n^\varepsilon(t))\, dW_n(t) , u_n^\varepsilon(t)) \to (12)$$

Define , $\tau_N = \inf\{t : |u_n^\varepsilon(t)|^2 + \int_0^t \|u_n^\varepsilon(s)\|^2\, ds > N\}$

$$|u_n^\varepsilon(t\wedge\tau_N)|^2 + 2v\int_0^{t\wedge\tau_N}\|u_n^\varepsilon(s)\|^2\, ds$$

$$= |u_n^\varepsilon(0)|^2 + 2\int_0^{t\wedge\tau_N}\left(\frac{1}{v}|f(s)|_{v'}^2 + \frac{v}{4}\|u_n^\varepsilon(s)\|^2\right) ds + 2\sqrt{\varepsilon} + \int_0^{t\wedge\tau_N} tr(\sigma_n(s,u_n^\varepsilon(s))Q\sigma_n(s,u_n^\varepsilon(s)))\, ds$$

$$\int_0^{t\wedge\tau_N}(\sigma_n(s,u_n^\varepsilon(s))\, dW_n(s) , u_n^\varepsilon(s)) \to (13)$$

Taking the expectation on both sides , and using the given condition

$$E\, |u_n^\varepsilon(t\wedge\tau_N)|^2 + E\frac{3}{2}v\int_0^{t\wedge\tau_N}\|u_n^\varepsilon(s)\|^2\, ds$$

$$\le E|u_n^\varepsilon(0)|^2 + \frac{2}{v}E\int_0^{t\wedge\tau_N}|f(s)|^2\, ds$$

$$+ \varepsilon KE\int_0^{t\wedge\tau_N}(1+\|u_n^\varepsilon(s)\|^2)\, ds$$

If $\varepsilon < \frac{v}{2k}$ , then

$$E\,(|u_n^\varepsilon(t\wedge\tau_N)|^2) + vE\int_0^{t\wedge\tau_N}\|u_n^\varepsilon(s)\|^2\, ds \le E|u(0)|^2 + \frac{2}{v}E\int_0^{t\wedge\tau_N}|f(s)|_{v'}^2\, ds + \varepsilon KT \to (14)$$

Taking the supremum up to time $T\wedge\tau_N$ in equation (13) and then taking the expectation ,

$$E\sup_{0\le t\le T\wedge\tau_N}(|u_n^\varepsilon(t)|^2 + v\int_0^t\|u_n^\varepsilon(s)\|^2\, ds)$$

$$\le E|u(0)|^2 + \frac{2}{v}E\int_0^{T\wedge\tau_N}|f(s)|_{v'}^2\, ds + \varepsilon KT$$

$$+2\sqrt{\varepsilon E}\sup_{0\le t\le T\wedge\tau_N}\left|\int_0^t\left(\sigma_n(s,u_n^\varepsilon(s))\, dW_n(s), u_n^\varepsilon(s)\right)\right| \to (15)$$

Again using the condition $|\sigma(t,u)|_{L_Q}^2 \le K(1+\|u\|^2)$ and then the Cauchy – Schwartz inequality

$$2\sqrt{\varepsilon E}\sup_{0\le t\le T\wedge\tau_N}\left|\int_0^t\left(\sigma_n(s,u_n^\varepsilon(s))\, dW_n(s), u_n^\varepsilon(s)\right)\right| \le \sqrt{2\varepsilon}2KE\{(\int_0^{T\wedge\tau_N}(1+\|u_n^\varepsilon(s)\|^2|u_n^\varepsilon(s)|^2)\, ds)^{1/2}\}$$

$$\le \sqrt{2\varepsilon}\ K\,(E\sup_{0\le t\le T\wedge\tau_N}|u_n^\varepsilon(t)|^2 + E\int_0^{T\wedge\tau_N}\|u_n^\varepsilon(s)\|^2\, ds + T)$$

Using the above equation in (15) , we get

$$E\sup_{0\le t\le T\wedge\tau_N}|u_n^\varepsilon(t)|^2 \le E|u(0)|^2$$

$$+ \frac{2}{v}\int_0^T E\,|f(s)|_{v'}^2\, ds + (\varepsilon K + \sqrt{2\varepsilon})T$$

$$+ \sqrt{2\varepsilon}\ K\,(E\sup_{0\le t\le T\wedge\tau_N}|u_n^\varepsilon(t)|^2 + E\int_0^{T\wedge\tau_N}\|u_n^\varepsilon(s)\|^2\, ds) \to (16)$$

From the equation (14) , it is easy to see that

$$v\, E\int_0^{T\wedge\tau_N}\|u_n^\varepsilon(s)\|^2\, ds \le E|u(0)|^2$$

$$+ \frac{2}{v}\int_0^T E\,|f(s)|_{v'}^2\, ds + \varepsilon KT \to (17)$$

Using the above equation in (16) , and that $2\varepsilon K < 1$

$$E\sup_{0\le t\le T\wedge\tau_N}|u_n^\varepsilon(t)|^2 \le C\,(E|u(0)|^2 , \int_0^T E\,|f(s)|_{v'}^2\, ds , v , T) \to (18)$$

The inequalities (14) and (18) we get ,

$$E\sup_{0\le t\le T\wedge\tau_N}|u_n^\varepsilon(t)|^2 + v\int_0^{T\wedge\tau_N}\|u_n^\varepsilon(s)\|^2\, ds$$

$$\le C(E|u(0)|^2 , \int_0^T E\,|f(s)|_{v'}^2\, ds , v , T) \to (19)$$

The equation (19) shows that $T\wedge\tau_N$ increases to T almost surely as $N\to\infty$. Taking the limit in equation (19) as $N\to\infty$ we get ,

$$E(\sup_{0\le t\le T}|u_n^\varepsilon(t)|^2 + v\int_0^T\|u_n^\varepsilon(s)\|^2\, ds)$$

$$\le C(E|u(0)|^2 , E\int_0^T|f(s)|_{v'}^2\, ds , v , T)$$

Next to prove,

$$E(|u_n^\varepsilon(t)|^2)e^{-\delta t} + v\int_0^t E\|u_n^\varepsilon(s)\|^2 e^{-\delta s} ds$$

$$\le E|u(0)|^2 + \frac{1}{\delta}E\int_0^t|f(s)|_{v'}^2\, ds + \frac{\varepsilon K}{\delta}e^{-\delta t}$$

$$(d|u_n^\varepsilon(t)|^2 + \delta|u_n^\varepsilon(t)|^2 dt + 2v\|u_n^\varepsilon(t)\|^2\, dt)$$

$$= e^{-\delta t}(2\langle f(t), u_n^\varepsilon(t)\rangle + \varepsilon tr$$

$$(\sigma_n(t,u_n^\varepsilon(t))Q\sigma_n(t,u_n^\varepsilon(t)))dt + 2\sqrt{\varepsilon}\ e^{-\delta t}(\sigma_n(t,u_n^\varepsilon(t))\, dW_n(t) , u_n^\varepsilon(t)) \to (20)$$

Define ,

$$\tau_N = \inf\{t : \max(|u_n^\varepsilon(t)|^2, \int_0^t\|u_n^\varepsilon(s)\|^2\, ds) > N\}$$

The equation (20) in integral form up to time $t\wedge\tau_N$ , taking expectations

$$E|u_n^\varepsilon(t\wedge\tau_N)|^2 e^{-\delta t\wedge\tau_N} + 2\, v\, E\int_0^{t\wedge\tau_N} e^{-\delta s}\,\|u_n^\varepsilon(s)\|^2\, ds$$

$$\le E|u_n^\varepsilon(0)|^2 + \frac{1}{\delta}E\int_0^{t\wedge\tau_N} e^{-\delta s}\,|f(s)|^2\, ds$$

$$+ \varepsilon E\int_0^{t\wedge\tau_N} tr(\sigma_n(s,u_n^\varepsilon(s))Q\sigma_n(s,u_n^\varepsilon(s)))\, ds$$

$$\le E|u_n^\varepsilon(0)|^2 + \frac{1}{\delta}E\int_0^{t\wedge\tau_N} e^{-\delta s}\,|f(s)|_{v'}^2\, ds$$

$$+ \varepsilon KE\int_0^{t\wedge\tau_N} e^{-\delta s}\,(1+\|u_n^\varepsilon(s)\|^2)ds$$

If $< \frac{v}{k}$ , the above equation we get ,

$$E|u_n^{\varepsilon}(t\wedge\tau_N)|^2\, e^{-\delta t\wedge\tau_N} + v\, E \int_0^{t\wedge\tau_N} e^{-\delta s}\, \|u_n^{\varepsilon}(s)\|^2\, d \le E|u_n^{\varepsilon}(0)|^2 + \frac{1}{\delta} E \int_0^{t\wedge\tau_N} e^{-\delta s}\, |f(s)|_{v'}^2\, ds + \frac{\varepsilon K}{\delta} \quad \to (21)$$

The above inequality as $\tau_N \to \infty$ almost surely we get ,

$$E(|u_n^{\varepsilon}(t)|^2)e^{-\delta t} + v\int_0^t E\|u_n^{\varepsilon}(s)\|^2\, e^{-\delta s} ds \le E|u(0)|^2 + \frac{1}{\delta} E\int_0^t |f(s)|_{v'}^2\, ds + \frac{\varepsilon K}{\delta}.$$

Next to prove ,

$$E(|u_n^{\varepsilon}(t)|^4)e^{-\delta t} + 3v\int_0^t E\|u_n^{\varepsilon}(s)\|^2\ |u_n^{\varepsilon}(s)|^2 e^{-\delta s} ds \le E|u(0)|^4 + C_\delta E\int_0^t |f(s)|_{v'}^4\, e^{-\delta s} ds + \frac{4\varepsilon K}{\delta} \sup_{0\le t\le T} E(|u_n^{\varepsilon}(s)|^2$$

Taking the expectation and using the condition

$|\sigma(t,u)|_{L_Q}^2 \le K(1+\|u\|^2)$ as before , we one obtains

$$Ee^{-\delta t}\ |u_n^{\varepsilon}(t)|^4 + 4v\int_0^t E\|u_n^{\varepsilon}(s)\|^2\ |u_n^{\varepsilon}(s)|^2 e^{-\delta s} ds$$

$$\le E|u_n^{\varepsilon}(0)|^4 + C_\delta E\int_0^t |f(s)|_{v'}^4\, e^{-\delta s} ds$$

$$+ 2\ \varepsilon K E\int_0^t (1+\ \|u_n^{\varepsilon}(s)\|^2)|u_n^{\varepsilon}(s)|^2\, e^{-\delta s} ds + 2\varepsilon K E\int_0^t |u_n^{\varepsilon}(s)|^2(1+\ \|u_n^{\varepsilon}(s)\|^2)\, e^{-\delta s} ds$$

From equation (8) , there exists a constant $M_T$ such that $\sup_{t\in[0,T]} E|u_n^{\varepsilon}(t)|^2 < M_T$

If $4\varepsilon K < v$ , the above inequality we get ,

$$E(|u_n^{\varepsilon}(t)|^4)e^{-\delta t} + 3v\int_0^t E\|u_n^{\varepsilon}(s)\|^2\ |u_n^{\varepsilon}(s)|^2 e^{-\delta s} ds$$

$$\le E|u_n^{\varepsilon}(0)|^4 + C_\delta E\int_0^t |f(s)|_{v'}^4\, e^{-\delta s} ds$$

$$+ 4\varepsilon K E\int_0^t |u_n^{\varepsilon}(s)|^2\, e^{-\delta s} ds$$

$$\le E|u_n^{\varepsilon}(0)|^4 + C_\delta E\int_0^t |f(s)|_{v'}^4\, e^{-\delta s} ds + \frac{4\varepsilon K M_T}{\delta}$$

$$E(|u_n^{\varepsilon}(t)|^4)e^{-\delta t} + 3v\int_0^t E\|u_n^{\varepsilon}(s)\|^2\ |u_n^{\varepsilon}(s)|^2 e^{-\delta s} ds \le E|u(0)|^4 + C_\delta E\int_0^t |f(s)|_{v'}^4\, e^{-\delta s} ds + \frac{4\varepsilon K}{\delta} \sup_{0\le t\le T} E(|u_n^{\varepsilon}(s)|^2$$

**4.0 The large deviations for navier– stokes equations**

**4.1 Theorem**

Let M be any fixed finite positive number .

$K_M = \{u_v \in C([0,T]: H) \cap L^2(0,T : V) : v \in S_M\}$

Where $u_v$ is the unique solution in $C([0,T]: H) \cap L^2(0,T : V)$ of the equation

$u_v(t) + [vAu_v(t) + B(u_v(t))]dt =$ [f(t) + $\sigma(t, u_v(t))v(t)$]dt.With $u_v(0) = \xi \in H$.Then $K_M$ is compact in $C([0,T]: H) \cap L^2(0,T : V)$ .

**Proof :**

Let $\{u_n\}$be a sequence in $K_M$ where $u_n$ corresponds to the solution of (2) with $v_n \in S_M$ in place of v . By weak compactness of $S_M$ , there exists a subsequence of $\{v_n\}$ which converges to a limit v weakly in $L^2(0,T:H_0)$ .

The subsequence is indexed by n. Define u as the solution of the equation

U(t) + [vAu(t) + B(u(t))] dt = [f(t) + $\sigma(t,u(t))v(t)$] dt → (3)

By the energy equality ,

$$|u(t)|^2 + 2v\int_0^t \|u(s)\|^2 ds = |\xi|^2 + 2\int_0^t \{\langle f(s), u(s)\rangle + (\sigma(s,u(s)v(s), u(s))\}$$

$$\le |\xi|^2 + 2\int_0^t \{|\sigma(s,u(s))v(s)| + |f(s)|_{v'}\}\, \|u(s)\|\, ds$$

$$\le |\xi|^2 + \frac{1}{v}\int_0^t |f(s)|^2\, ds v\int_0^t |u(s)|^2\, ds + \int_0^t |\sigma(u(s))|_{L_Q}\, |v(s)|_0 |u(s)| ds$$

Using the given condition

$$|u(t)|^2 + v\int_0^t \|u(s)\|^2 ds \le |\xi|^2 + \frac{1}{v}\int_0^t |f(s)|_{v'}^2\, ds + \frac{v}{2}\int_0^t (1+\|u(s)\|^2) ds + \frac{K}{v}\int_0^t |v(s)|_0 |u(s)|^2\, ds \quad \to (4)$$

By the Gronwall lemma ,

For any T,

$$\sup_{0\le t\le T} |u(t)|^2 \le (|\xi|^2 + \frac{1}{v}\int_0^T |f(s)|_{v'}^2\, ds)\, e^{\frac{k}{v}\int_0^T |v(s)|_0^2 ds} \quad \to (5)$$

Using the above bound in the equation (4)

It follows that ,

$$\int_0^T \|u(s)\|^2 ds \le C(v, |\xi|^2, \int_0^T |f(s)|_{v'}^2\, ds, M, K, \sup_{0\le t\le T} |u(s)|^2) \to (6)$$

Thus , $\sup_{0\le t\le T} |u(t)|^2 + \int_0^T \|u(t)\|^2 ds$

$$\le C_1(v, |\xi|^2, \int_0^T |f(s)|_{v'}^2\, ds, M, K, \sup_{0\le t\le T} |u(s)|^2) \to (7)$$

By the equation (5) , the above bound can be written as $\sup_{0\le t\le T} |u(t)|^2 + \int_0^T \|u(t)\|^2 ds$

$$\le C_2(v, |\xi|^2, \int_0^T |f(s)|_{v'}^2\, ds, M, K,) \quad \to (8)$$

So that the bound is uniform in n . Let $w_n = u_n - u$ . It sufficies to show that $w_n \to 0$ in $C([0,T]: H) \cap L^2(0,T : V)$ as n $\to \infty$

$$w_n(t) + \int_0^t \{vAw_n(s) + B(u_n(s)) - B(u(s))\} ds$$

$$= \int_0^t \{\sigma(s, u_n(s))v_n(s) - \sigma(s, u(s))v(s)\} ds$$

So that ,

$\frac{1}{2}|w_n(t)|^2 + v\int_0^t \|w_n(s)\|^2 ds$

$+\int_0^t \{b(u_n(s), u_n(s), w_n(s)) - \qquad b(u(s), u(s), w_n(s))\}$

$= \int_0^t \{((\sigma(s, u_n(s)) - \sigma(s, u_n(s)))\, v_n(s), w_n(s))$

+s($(\sigma(s,u(s))(v_n(s) - v(s)), w_n(s))$-}ds → (9)

Define b( .,.,.) : $V \times V \times V \to \mathcal{R}$ by

b(u,v,w) = $\sum_{i,j=1}^{2} \int_G u_i \frac{\partial v_j}{\partial x_i} w_j\, dx$

b( $u_n, u_n, w_n$ ) – b(u , u , $w_n$ ) = b( $w_n, u_n, w_n$ ) + b(u , $w_n, w_n$) = b($w_n, u, w_n$)

So that ,

$|b(u_n, u_n, w_n) - b(u, u, w_n)| \le |w_n||w_n\|\|u\| \quad \to (10)$

The first term in the integrand on the ride side of the equation (9) can be bounded by L$\|w_n(s)\||w_n(s)||v_n(s)|_0$

The second term in the integrand of the equation(9) can be bounded by $|\sigma(s,u(s))(v_n(s) - v(s))|\,|w_n(s)|$. The above equations and equation(10) used in (9) we get ,

$\frac{1}{2}|w_n(t)|^2 + v\int_0^t \|w_n(s)\|^2 ds$

$\le \int_0^t |w_n(s)|\|w_n(s)\|\|u(s)\| ds$

$+\int_0^t L\|w_n(s)\||w_n(s)||v_n(s)|_0 ds$

$+ \int_0^t |\sigma(s)(v_n(s) - v(s))|\,|w_n(s)|$ds

$\frac{1}{2}|w_n(t)|^2 + \frac{v}{2}\int_0^t \|w_n(s)\|^2 ds$

$\le \frac{2(L+1)}{v}\int_0^t |w_n(s)|^2(\|u(s)\| + |v_n(s)|_0)^2$ds

$+ \int_0^t |\sigma(s,u(s))(v_n(s) - v(s))|^2\, ds$

So that the Gronwall inequality

$\frac{1}{2}|w_n(t)|^2 + \frac{v}{2}\int_0^t \|w_n(s)\|^2 ds$

$\le (\int_0^T |\sigma(s,u(s))(v_n(s) - v(s))|^2\, ds \quad )$

$e^{\frac{2(L+1)}{v}\int_0^t (\|u(r)\| + |v_n(r)|_0)^2 dr}$

Since $\sigma(.,.)Q^{1/2}$ is a Hilbert-schmidt operator on H , and hence a compact operator on H . Using the equation (8) and the weak convergence of $v_n \to v$ in $S_M$ we obtain ,

$\sup_{0\le t\le T} |w_n(t)|^2 + \int_0^T \|w_n(s)\|^2 ds \to 0$ as n → $\infty$

$\therefore$ $K_M$ is compact in $C([0,T]:H) \cap L^2(0,T:V)$

**Conclusion**

In this work, we discussed about the transformation of Navier-stokes equations into a system of functional integrals and also Girsanov theorem, Burkholder-Davis-Gundy inequality. Finally, stochastic Navier-stokes equations and large deviations results .

**References**

**[1] Ben Artzi M (1994)** global solutions of two-dimensional navier stokes and euler equations.

**[2] Barbara Busnello (1999**) A probabilistic approach to the two dimensional Navier-stokes equations.

**[3] Daniel Revuz and Marc Yor(1994**) Continuous martingales and Brownian motion.

**[4] Menaldi JL and Sritharan SS (2002)** Stochastic two dimensional Navier-stokes equations.

**[5] Prakash Balachandran (2008)** Stochastic integration, 2008.

**[6] Sritharan SS and Sundar P (2006)** Large deviations for the two dimensional Navier-stokes equations with multiplicative noise.

**[7] Chang MH (1996)** Large deviation for Navier–Stokes equations with small stochastic perturbation.

**[8] Varadhan (1984)** Large Deviations and its Applications.

**Citation :** Kiruthika R and Ramadevi S (2014) Navir – Stokes Equation For two Dimentional In Fluid Dynamics, *Int J Adv Interdis Res*, 1(9): 12-19.

**Received :** August 16,2014 | **Revised :** August 20,2014 | **Accepted :** August 22,2014 | **Published:** August 26,2014