Original Research Article Open Access

Vorticity and Navier – Slip boundary conditions in 3D Lagrangian Navier – stokes α model

Kavitha S and Jeevitha K*

Department of Mathematics, Vivekanandha College of Arts & Sciences for women(Autonomous), Tiruchengode, Namakkal,Tamil Nadu, India.

Abstract

Vorticity – slip boundary conditions and the Navier – slip boundary conditions in 3 Dimensional Lagrangian Navier – stokes α model and the limiting Navier stokes system on smooth bounded domains. It establishes the spectrum properties and regularity estimates of the associated Stokes operators, the local well – posedness of the strong solution and global existence of weak solutions for initial boundary value problems for such systems. Furthermore, the vanishing α limit to a weak solution of the corresponding initial boundary value problem of the Navier stokes system is proved and a rate of convergence is shown for the strong solution.

Keywords Navier – stokes α model, Navier – slip boundary conditions, vanishing α limit, weak solution, strong solution.

INTRODUCTION

The Lagragian Navier – Stokes α model (LNS - α) as a regularization system of the Navier – Stokes equations (NS) is given by

$$\partial_t v - \Delta v + T_\alpha v.\nabla v + \nabla (T_\alpha v)^T.v + \nabla p = 0$$
$$\nabla.v = 0$$

We consider the vanishing viscosity limit problem from the Navier–Stokes flows on a general 3-dimensional bounded domain with Navier-slip boundary condition. The viscous flow is governed by

$$\partial_t u^v - v\Delta u^v + (u^v.\nabla)u^v + \nabla \pi^v = 0$$
$$\nabla.u^v = 0$$

with the boundary conditions

$$u^v.\vec{n} = 0, \text{ (curl } u^v)\times \vec{n} = 0$$

In the absence of physical boundaries, then any smooth solutions to the Euler system can be approximated by ones to Navier–Stokes equations. Navier first proposed the slip boundary condition i.e. the tangential velocity proportional to the tangential component of the viscous stress while maintaining the no-flow condition in the normal direction, which is now called Navier boundary condition. This boundary condition was rigorously justified as the effective boundary conditions for flows over rough boundaries.

For the general Navier-slip conditions, obtained that the solution u^v converges to the solution u^0 of Euler equations in $L^\infty(0,T;L^2(\mathbb{R}^2_+))$ assuming that initial vorticity is uniformly bounded.

We observed that in both dimension two and three a direct L^2 estimate yields the strong L^2 convergence to Euler equation and that the convergence in H^2 is impossible in general. Main motivation for the proposed VSB that the vorticity formulations of the fluid equations have played important roles in analyzing fluid motions.

*Address Correspondence at Department of Mathematics, Vivekanandha College of Arts & Sciences for women (Autonomous), Tiruchengode, Namakkal, Tamil Nadu, India. Ph:+8883086477 Email: jeevithamathsca@gmail.com

we apply Hodge Decomposition theory to the Stokes problems with both VSB and NSB conditions. First we consider, a special Stokes problem with VSB and Next topology of the domain is assumed to be general, to avoid uniqueness of the solutions for the general Stokes problems and consider perturbed Stokes problem associated with VSB and we prove the associated Stokes operator is a self – adjoint extension of the associated positive definite bilinear form. Next problem can be used to prove well – posedness of the non – homogeneous and boundary value problem by construction a contraction map. Finally we identify the relationship between NSB and VSB.

For Navier – slip conditions with fixed slip length (α = const) and they improved the strong L^2 convergence with the rate $O(v^{\frac{3}{4}})$. For the special case of flat boundaries, obtained the uniform H^2 - convergence theory with the optimal convergence rate. Finally, we also obtain the $W^{1,p}$- estimates with $3 < p \le 6$ and prove the convergence in $L^\infty((0, T) \times \Omega)$ with an optimal rate of order $O(v^{\frac{1}{2}})$.

1.Basic Notations:

Ω is a bounded or unbounded smooth domain of the three – dimensional space $\mathbb{R}^3$. $H^3(\Omega)$ denote the standard Hilbert space. β be nonnegative smooth function.

2. Hodge Decomposition and its properties:

The Hodge Decomposition theory is an important role in 3D bounded smooth domain and give simple L^2 version below.

Theorem 2.1: Prove that $L^2(\Omega)$ = DF ⊕ GG

Proof:

$$\text{DF} = \{ u \in L^2(\Omega); \nabla.u = 0 \}$$
$$\text{GG} = \{ u \in L^2(\Omega); u = \nabla\varphi, \varphi \in H_0^1(\Omega)\}$$

Let $u \in$ DF.

Then $u.n$ is well-defined on $\partial\Omega$ and

$$\int_{\partial\Omega} u.n = \int_{\Omega} \nabla.u = 0$$

Let φ solve

$$\Delta\varphi = 0 \text{ in } \Omega \quad \text{and} \quad \partial_n\varphi = u.n \text{ on } \partial\Omega$$

Combining we get,

DFG = $\{u \in L^2(\Omega);\ u = \nabla\varphi, \nabla.u = 0, \int_{\partial\Omega} u.n = 0\}$

Hence $v \in H$.

Assume that

$v = u - \nabla\varphi$ and

H = $\{u \in L^2(\Omega);\ \nabla.u = 0$, in Ω; $u.n = 0$; on $\partial\Omega\}$

$$(u, v) = 0, \forall\, u \in H, v \in DFG.$$

It follows further decomposition,

To prove that DF = H ⊕ DFG

Take, $u = \nabla\varphi \in$ DFG and range of curl

$\nabla \times H^1(\Omega)$ is closed in $L^2(\Omega)$.

DFG further decomposed to DFG = CG ⊕ HG where

CG = DFG ∩ ($\nabla \times H^1(\Omega)$), HG = DFG ∩ $(\nabla \times H^1(\Omega))^{\perp}$

Let $u = \nabla\varphi \in$ HG.

Since,

$$0 = ((\nabla \times v), \nabla\varphi) = \int_{\partial\Omega} (n \times v).\nabla\varphi$$

for all $v \in H^1(\Omega)$, thus $\partial_\tau\varphi = 0$, on $\partial\Omega$ with τ being any tangential direction on $\partial\Omega$ which implies φ is a constant on each component Γ_i of $\partial\Omega$.

Then HG = $\{\nabla\varphi;\ \Delta\varphi = 0, \varphi = c_i$ on $\Gamma_i\}$

It consists only smooth vectors and finite dimensional, which is called the Harmonic Gradient Space.

CG can be expressed as

CG = $\{u \in L^2(\Omega);\ u = \nabla\varphi, \nabla.u = 0, \int_{\Gamma_i} u.n = 0\}$

Since CG ⊂ $\Delta \times H^1(\Omega)$, here used in curl type gradient space.

Take HH = H ∩ Ker ($\nabla \times$)

Then

HH = $\{u \in L^2(\Omega);\ \nabla.u = 0, \nabla \times u = 0$ in Ω , $u.n = 0$; on $\partial\Omega\}$

This is called Harmonic Knots Space, which consists only smooth functions and finite dimensional.

Now, H can be decomposed to

$$\text{H} = \text{FH} \oplus \text{HH}$$

where FH = H ∩ $(\text{Ker}\,(\nabla \times))^{\perp}$

In conclusion, we have

$L^2(\Omega) = \text{FH} \oplus \text{HH} \oplus \text{CG} \oplus \text{HG} \oplus \text{GG}$

Its uniquely written as

$$u = P_{HH}u + P_{FH}u + P_{CG}u + P_{HG}u + P_{GG}u$$

where, P_X denotes the projection on the corresponding subspace.

The space FH can be expressed as

FH = $\{u \in L^2(\Omega);\ \nabla.u = 0,\ u.n = 0$; on $\partial\Omega$, F$(u) = 0\}$

FH = $\{u; u = \nabla \times v, v \in H^1(\Omega), \nabla.v = 0, n \times v = 0$ on $\partial\Omega\}$

where F$(u) = 0$ means $\int_\Sigma u.n = 0$

for any smooth cross section Σ of Ω.

To fact that HH ⊂ $\nabla \times$ (FH ∩ $H^1(\Omega)$)

such that $L^2(\Omega) = \nabla \times (\text{FH} \cap H^1(\Omega)) \oplus \text{HG} \oplus \text{GG}$ and

$H^s(\Omega) = \nabla \times (\text{FH} \cap H^{s+1}(\Omega)) \oplus (\text{HG} \cap H^s(\Omega)) \oplus (\text{GG} \cap H^s(\Omega))$

for $s \geq 0$.

$C^\infty(\Omega) \cap$ X is dense in $H^s(\Omega) \cap$ X, $s \geq 0$

By conclusion, X = FH, HH, CG, HG, GG.

Therefore, HH and HG are finite dimensional.

3.A Special Stokes Problem:

Consider the following special Stokes problem with homogeneous VSB condition

$$-\Delta u = f \text{ in } \Omega$$

$$\nabla.u = 0 \text{ in } \Omega$$

$$u.n = 0, n \times \nabla \times u = 0 \text{ on } \partial\Omega \quad \text{with } f \in FH.$$

Let W = $\{u \in H^2(\Omega);\ n \times (\nabla \times u) = 0$ on $\}$

Theorem 3.1: The Stokes operator $A_F = -\Delta$ with the domain D(A_F) = W ∩ FH is self – adjoint in the Hilbert space FH.

Proof:

$C_0^\infty(\Omega) \cap$ H is dense in H. It follows that A_F is densely defined due to the orthogonality of FH and HH and the compactness of HH.

Let $u \in$ W.

Since $n \times (\nabla \times u) = 0$ on $\partial\Omega$,

Then, $-\Delta u = \nabla \times (\nabla \times u) \in FH$

FH = $\{u \in L^2(\Omega);\ \nabla.u = 0,\ u.n = 0$; on $\partial\Omega$, F$(u) = 0\}$

FH = $\{u; u = \nabla \times v, v \in H^1(\Omega), \nabla.v = 0, n \times v = 0$ on $\partial\Omega\}$

where F$(u) = 0$ means

$$\int_\Sigma u.n = 0$$

for any smooth cross section Σ of Ω.

That there is $\phi \in H^1(\Omega)$, satisfying

$\nabla \times \phi = f$ in Ω

$\nabla.\phi = 0$ in Ω

$\phi \times n = 0$ on $\partial\Omega$

By using theorem 2.10, there is $v \in FH \cap H^2(\Omega)$

Then $\phi = \nabla \times v + P_{HG}\,\phi$ and $P_{HG}\,\phi \times n = 0$ on $\partial\Omega$

It follows that $n \times (\nabla \times v) = 0$ on $\partial\Omega$

Then $\nabla \times (P_{HG}\,\phi) = 0$ and implies that $-\Delta v = f$ in Ω

Thus $A_F: W \cap FH \to FH$ is surjective.

If $f = 0$, then integration by part shows $\|\nabla \times v\| = 0$

It follows that $u = 0$ due to the orthogonality of FH and HH then,

$A_F: W \cap FH \to FH$ is one to one.

W and FH are closed in $H^2(\Omega)$ and $L^2(\Omega)$, and $\|\Delta v\| \leq \|v\|_2$

We obtain from the Banach inverse operator theorem that

$\|v\|_2 \leq c\|\Delta v\|$

The theorem was proved.

Result:

Let $u \in H_n^1(\Omega)$ and $u \in FH \cap H^1(\Omega)$. Then the following Poincare type inequality holds $||u|| \leq c\,||\nabla \times u||$.

Theorem 3.2: The operator $A_F = -\Delta$ with the domain D(A_F) = W ∩ FH is self – adjoint in the Hilbert space FH of bilinear form

$a(u, \phi) = (\nabla \times u, \nabla \times \phi), D(a)$

$$V_F = FH \cap H^1(\Omega)$$

Proof:

$a(u, \phi)$ is closed and positive.

It follows that there is a self – adjoint operator A with domain D(A) ⊂ D(a) such that

$a(u, \phi) = (A\,u, \phi), \forall\, \phi \in$ FH ∩ $H^1(\Omega)$

for any $u \epsilon$ D(A).

It is clear that D(A_F) = W ∩ FH ⊂ D(A) and

$Au = -\Delta u \; \forall \; u \; \epsilon$ W ∩ FH.

Let $u \epsilon$ D(A) and $f = Au$.

It follows that $f \epsilon FH$ and $v \; \epsilon$ D(A_F) such that

$-\Delta u = f$ in Ω

$\nabla . u = 0$ in Ω

$u . n = 0, n \times \nabla \times u = 0$ on $\partial \Omega$ with $f \epsilon FH$.

These equations are valid with u replaced by v and $a(v, \phi)$ = (f, ϕ).

On the other hand,

$a(u, \phi)$ = (A u, ϕ) = (f, ϕ) for all $\phi \; \epsilon \; V_F$.

Hence, $a(u - v, \phi)$ = $(\nabla \times (u - v), \nabla \times \phi)$ = 0 for all $\phi \; \epsilon \; V_F$.

Taking, $\phi = u - v$ shows that $\nabla \times (u - v)$ = 0.

Thus, $u = v$.

Hence, D(A) = D(A_F) and A = A_F.

3.3The Stokes problem with VSB condition:

Consider the Stokes problem with general VSB condition. Since the domain is allowed to have general topology, the kernel of $-\Delta$ may not be empty.

Consider the following boundary value problem,

$(I - \Delta)u + \nabla p = f$ in Ω

$\nabla . u = 0$ in Ω

$u . n = 0, n \times (\nabla \times u) = \beta u$ on $\partial\Omega$

where β is a non negative smooth function.

Define V = $H^1(\Omega)$ ∩ H

$W_\beta = \{u \; \epsilon \; H^2(\Omega); n \times (\nabla \times u) = \beta u \;$ on $\partial\Omega\}$

Define a bilinear form as

$$\tilde{\alpha}_\beta(u, \phi) = (u, \phi) + \alpha_\beta(u, \phi)$$

where

$$\alpha_\beta(u, \phi) = \int_{\partial\Omega} \beta u . \phi + \int_{\Omega} (\nabla \times u).(\nabla \times \phi)$$

with the domain D($\tilde{\alpha}_\beta$) = V.

$u \epsilon$ V is said to be weak solution to the boundary value problem by using equation on H for $f \epsilon V'$ if $\tilde{\alpha}_\beta(u, \phi)$ = $(f, \phi), \forall \phi$ in V. where V' is the dual space of V.

Theorem 3.3: Let $\epsilon H^{-\frac{1}{2}}(\partial\Omega)$, $b.n$ = 0. Then for any $f \epsilon V'$, the boundary value problem

$\lambda u - \Delta u + \nabla p = 0$ in Ω

$\nabla . u = 0$ in Ω

$u . n = 0, n \times (\nabla \times u) = b$ on $\partial\Omega$

has a unique weak solution $u \epsilon$ V in the sense of

$$\lambda(u, \phi) + \int_{\Omega} (\nabla \times u).(\nabla \times \phi) + \int_{\partial\Omega} b.\phi = (f, \phi), \forall \phi \; \epsilon \; V$$

Proof:

Definition of the weak tangential trace, $H^{-\frac{1}{2}}(\partial\Omega)$. For any given smooth and nonnegative function β, we define a map

$$T : H^{\frac{1}{2}}(\Omega) \cap H \mapsto V \subset H^{\frac{1}{2}}(\Omega) \cap \text{H}$$

By $u = Tv$ determined and b replaced by $\beta v + b$ and $f = 0$.

Let $v_i \in H^{\frac{1}{2}}(\Omega)$ ∩ H and $u_i = Tv_i$, i = 1,2. It follows that

$$\lambda \parallel u_1 - u_2 \parallel^2 + \parallel \nabla \times (u_1 - u_2) \parallel^2 + \int_{\partial\Omega} \beta(u_1 - u_2).(v_1 - v_2) = 0$$

Taking modulus,

$$\left| \int_{\partial\Omega} \beta(u_1 - u_2).(v_1 - v_2) \right| \leq \text{c} \|u_1 - u_2\|_{H^{\frac{1}{2}}(\Omega)} \|v_1 - v_2\|_{H^{\frac{1}{2}}(\Omega)}$$

and

$$\|\varphi\|^2_{H^{\frac{1}{2}}(\Omega)} \leq c\|\varphi\|\|\varphi\|_{H^1(\Omega)} \leq c\|\varphi\|\|\nabla \times \varphi\|, \forall \; \varphi \; \epsilon \; \text{V}$$

It follows that

$\|u_1 - u_2\|_{H^{\frac{1}{2}}(\Omega)} \leq c\lambda^{-\frac{1}{2}}\|v_1 - v_2\|^2_{H^{\frac{1}{2}}(\Omega)}$ for $\lambda \geq 1$.

Take λ large enough such that T becomes a contraction map on $H^{\frac{1}{2}}(\Omega)$. It follows that $Tv = v$ has a unique solution ψ on $H^{\frac{1}{2}}(\Omega)$ and ψ = Tψ $\epsilon \; H^1(\Omega)$. For any $\tilde{f} \in V'$, let v be the weak solution with $f = \tilde{f} - (1 - \lambda)\psi$. It is clear that $u = v + \psi \; \epsilon$ V is a weak solution of the following problem:

$u - \Delta u + \nabla p = \tilde{f}$ in Ω

$\nabla . u = 0$ in Ω

$u . n = 0$

$n \times (\nabla \times u) = \beta u + b$ on $\partial\Omega$

In the sense that

$$(u, \phi) + \int_{\Omega} (\nabla \times u).(\nabla \times \phi) + \int_{\partial\Omega} (\beta u + b).\phi = (\tilde{f}, \phi),$$

for all ϕ in V. Hence $u = v$.

Hence uniqueness proved

3.4 The Stokes problem with the NSB problem:

Consider the following Stokes problem with the NSB condition,

$(I - \Delta)u + \nabla p = f$ in Ω

$\nabla . u = 0$ in Ω

$u . n = 0, 2(S(u)n)_\tau = -\gamma u_\tau u$ on $\partial\Omega$

where γ is a nonnegative smooth function.

Define

$\widetilde{W_\gamma} = \{u \in H^2(\Omega); 2(S(u)n)_\tau = -\gamma \; u_\tau$ on $\}$

and a bilinear form

$\widetilde{a_\gamma}(u, \phi) = (u, \phi) + a_\gamma(u, \phi), D(\widetilde{a_\gamma}) = V$

where

$$a_\gamma(u, \phi) = \int_{\partial\Omega} \gamma . \phi + 2\int_{\Omega} S(u).S(\phi)$$

and S(u).S(ϕ) denotes the trace of the product of the two matrices. u is said to be a weak solution to the boundary value problem on H for $f \; \epsilon \; V'$ if $\widetilde{a_\gamma}(u, \phi) = (f, \phi)$, $\forall \; \phi$ in V. where V' is the dual space of V.

Lemma 3.4: Let $u \; \epsilon \; H^2(\Omega)$ and $u.n$ = 0 on the boundary. It holds that $(2(S(u)n) - \omega \times n)_\tau$ = GD$(u)_\tau$ with GD(u) = $-2S(n)u$.

Proof:

We have, $S(n)u = \frac{1}{2}\omega \times n + S(u)n$

$$\partial_n u = \frac{1}{2}\omega \times n + S(u)n$$

u replace τ,

$\partial_\tau u = \frac{1}{2}\omega \times \tau + S(u)\tau$

Adding above two equations, we have

$$2(S(u)n).\tau = \partial_\tau u.n + \partial_n u.\tau$$

Substracting above two equations, we have

$$(n \times \omega)\tau = \partial_\tau u.n - \partial_n u.\tau$$

$$2(S(u)n).\tau + (n \times \omega)\tau = 2\partial_\tau u.n$$

such that $u.n = 0$ on the boundary. It follows that

$$\partial_\tau u.n = -u.\partial_\tau n$$

we conclude that $(2S(u)n - \omega \times n).\tau = -2.\partial_\tau n,$

$$\partial_\tau n = \frac{1}{2}(\nabla \times n) \times \tau + S(n)\tau$$

Hence

$$(2S(u)n - \omega \times n).\tau = ((\nabla \times n) \times u).\tau - 2S(n)u.\tau$$

Let $u \times \tau = \lambda n$ and $(\nabla \times n).n = 0$ on the boundary. It follows that

$$(2S(u)n - \omega \times n).\tau = -2S(n)u.\tau$$

$$GD(u) = -2S(n)u$$

The lemma is proved.

4. Functional setting of the LNS – α Equation:

we formulate the following boundary value problem for the LNS – α system

$\partial_\tau v - \Delta v + \Delta \times v \times u + \nabla p = 0$ in Ω

$\nabla.v = 0$ in Ω

$u - \alpha\Delta u + \nabla\tilde{p} = v$ in Ω

$\nabla.u = 0$ in Ω

with the VSB conditions

$v.n = 0, n \times \nabla \times v = \beta v$ on $\partial\Omega$

$u.n = 0, n \times \nabla \times u = \beta u$ on $\partial\Omega$

Due to the self – adjoint extension of the bilinear form $\tilde{\alpha}_\beta(u,\phi)$ with the domain D$(\tilde{\alpha}_\beta) = V$ is Stokes operator $A_\beta = I + P(-\Delta)$ with D$(A_\beta) = W_\beta \cap H$, and A_β is an isomorphism between D(A_β) and H with compact inverse on H. consequently, the eigenvalues of the Stokes operator A_β can be listed as $1 \le 1 + \lambda_1 \le 1 + \lambda_2 \cdots \to \infty$ with the corresponding eigenvectors $\{e_j\} \subset W_\beta$,

i.e., $A_\beta e_j = (1 + \lambda_j)e_j$

which form a complete orthogonal basis in H.

Furthermore, it holds that

$(1 + \lambda_1) \| u \|^2 \le \tilde{\alpha}_\beta(u,u) \le \frac{1}{1+\lambda_1} \| A_\beta u \|^2$, $\forall\, u \in$ D(A_β)

$A_\alpha = I - \alpha P\Delta$ is also a positive definite self – adjoint operator with domain D$(A_\alpha) = W_\beta \cap H$ for any $\alpha > 0$.

Theorem 4.1: The nonlinearity $B_\alpha(v) : V \mapsto H$ is locally lipshitz for $\alpha > 0$.

Proof:

For any $v_1, v_2 \in V$,

$\| B_\alpha(v_1) - B_\alpha(v_2) \| \le \|\nabla \times (v_1 - v_2) \times T_\alpha v_1 - \nabla \times v_2 \times T_\alpha(v_1 - v_2) \|$

Note that for all $\phi \in$ V, $\psi \in L^\infty(\Omega)$,

$\| \nabla \times (\phi) \times \psi \| \le c \| \phi \|_1 \| \psi \|_{L^\infty(\Omega)}$ and

$$\| w \|^2_{L^\infty(\Omega)} \le c \| w \|_1 \| w \|_2$$

It follows that

$\|B_\alpha(v_1) - B_\alpha(v_2)\|$

$\le c(\| v_1 \|_1 + \| v_2 \|_1) \| v_1 - v_2 \|_1$

which implies the theorem.

4.2 Well – Posedness of the LNS – α Equations:

we investigate the well – posedness of the initial boundary value problem of the LNS – α equations by Gelerkin approximation based on the orthogonal basis. Now we see the theorem as Local Well – Posedness.

Corollary 4.2: Let $v_0 \in H$ and $\alpha > 0$. Then there is a time $T^* = T^*(v_0) > 0$ such that the problem has a unique weak solution of (v,u) with initial data v_0 on the interval $[0, T^*)$ in the sense of any T ϵ $(0, T^*)$, which satisfies the energy equation

$$\frac{d}{dt} \| v \|^2 + 2a_\beta(v,v) + (B(v,u),v) = 0, \text{on } [0,T]$$

in the sense of distribution. Furthermore, if $v_0 \in V$ then

$v \,\epsilon\, L^2(0,T;\, W_\beta \cap H) \cap C([0,T);V)$

$$v' \epsilon\, L^2(0,T;H)$$

and the energy equation

$$\frac{d}{dt} a_\beta(v,v) + 2 \| P\Delta v \|^2 + 2(B(v), -\Delta v) = 0$$

is valid.

4.3 Global Well – Posedness:

Theorem 4.3: If $v_0 \in H$, $\alpha > 0$, then the solution v obtained, Let $v_0 \in H$ and $\alpha > 0$. Then there is a time $T^* = T^*(v_0) > 0$ such that the problem has a unique weak solution of (v,u) with initial data v_0 on the interval $[0, T^*)$ in the sense of any $T \,\epsilon\, (0, T^*)$, which satisfies the energy equation

$$\frac{d}{dt} \| v \|^2 + 2\alpha_\beta(v,v) + (B(v.u),v) = 0, on\ [0,T]$$

in the sense of distribution. Furthermore, if $v_0 \in V$

Then $v \,\epsilon\, L^2(0,T;\, W_\beta \cap H) \cap C([0,T);V)$

$$v' \epsilon L^2(0,T;H)$$

and the energy equation

$$\frac{d}{dt}\alpha_\beta(v,v) + 2 \| P\Delta v \|^2 + 2(B(v), -\Delta v) = 0$$

is valid & its global.i.e., $T^* = T^*(v_0) = \infty$.

Proof:

Let v be the weak solution on the interval $[0,T]$. Then, $u = T_\alpha v \in L^\infty(0,T;\, W_\beta \cap H)$

Taking u as a test function we get

$(v',u) + a_\beta(v,u) + (B(v,u),u) = 0$

Since $v = (I - \alpha P\Delta)$,

Then $2(v',u) = \frac{d}{dt}(\| u \|^2 + \alpha a_\beta(u,u))$

in the sense of distribution on $[0,T]$.

we get

$$(B(v,u),u) = \int_\Omega (\nabla \times v) \times u.u = 0$$

It follows that

$$\frac{d}{dt}\left(\| u \|^2 + \alpha a_\beta(u,u)\right) + 2\Big(\int_{\partial\Omega} \beta u.v + \int_\Omega (\nabla \times v).(\nabla \times u)\Big) = 0$$

Due to the smoothness and the boundary condition for u, it holds that

$$\int_\Omega (\nabla \times v).(\nabla \times u) = -\int_{\partial\Omega} \beta u.v + \int_\Omega (-\Delta u).v$$

Consequently

$$\frac{d}{dt}\left(\| u \|^2 + \alpha a_\beta(u,u)\right) + 2\left(a_\beta(u,u) + \alpha \| P\Delta u \|^2\right) = 0$$

It follows that

$(\| u \|^2 + \alpha a_\beta(u,u)) \le (\| u_0 \|^2 + \alpha a_\beta(u_0,u_0))$

and

$$\int_0^t (a_\beta(u,u) + \alpha \| P\Delta u \|^2) d\tau \le (\| u_0 \|^2 + \alpha a_\beta(u_0,u_0))$$

On the other hand, it follows from the energy equation and a similar argument as for that

$$\frac{d}{dt} \| v \|^2 + a_\beta(v,v) \le \text{ c } \| v \|^4 + 1$$

Then $\| v \|^2 \le (\| u \|^2 + \alpha^2 \| P\Delta u \|^2$

It follows that $\| v \|^2 + \int_0^t a_\beta(v,v) \le c$

for some constant c depending only on v_0 and α. Thus $T^* = \infty$.

5. Vanishing *α* Limit and the NS Equations:

we investigate the vanishing *α* limit of the solutions of the LNS – *α* equations $(\alpha \to 0)$ to that of the NS equations. we will prove weak and strong convergence results. Then, the global existence of weak solutions and the local unique strong solution to the NS equations with the VSB condition are followed.

5.1 Weak Convergence and Global Weak Solutions of the NS:

Theorem 5.1: Let $v_0 \in H$, and (v^α, u^α) be the global weak solution.If $v_0 \in H$, $\alpha > 0$, then the solution v obtained, Let $v_0 \in H$ and $\alpha > 0$. Then there is a time $T^* = T^*(v_0) > 0$ such that the problem has a unique weak solution of (v,u) with initial data v_0 on the interval $[0, T^*)$ in the sense of any $T \epsilon (0, T^*)$, which satisfies the energy equation

$$\frac{d}{dt} \| v \|^2 + 2\alpha_\beta(v,v) + (B(v.u),v) = 0, on\ [0,T]$$

in the sense of distribution. Furthermore, if $v_0 \in V$

Then $v \,\epsilon\, L^2\left(0,T;\ W_\beta \cap H\right) \cap\ C([0,T);V)$

$$v' \epsilon L^2(0,T;H)$$

and the energy equation

$$\frac{d}{dt}\alpha_\beta(v,v) + 2 \| P\Delta v \|^2 + 2(B(v), -\Delta v) = 0$$

is valid and it is global, $T^* = T^*(v_0) = \infty$ corresponding to the parameter $\alpha > 0$ and given $T > 0$ there is a subsequence u^{α_j} of u^α and (v^0, u^0) satisfying

$$v^0 \in L^2(0,T;V) \cap C_w([0,T];H)$$

$$(v^0)' \in L^{\frac{4}{3}}(0,T;V')$$

such that

$v^{\alpha_j} \longrightarrow v^0$ in $L^2(0,T;H)$ weakly

$v^{\alpha_j} \longrightarrow v^0$ in $L^2(0,T;D(A_\beta^{-\frac{1}{4}}))$ strongly

$u^{\alpha_j} \longrightarrow v^0$ in $L^2(0,T;V_\beta)$ weakly

$u^{\alpha_j} \longrightarrow v^0$ in $L^2(0,T;D(A_\beta^{-\frac{1}{4}}))$ strongly

Moreover (v^0, v^0) is a weak solution of the initial boundary problem of the NS equations

$\partial_\tau v - \Delta v + \Delta \times v \times u + \nabla p = 0$ in Ω

$\nabla \,.\, v = 0$ in Ω

$u - \alpha\Delta u + \nabla\tilde{p} = v$ in Ω

$\nabla \,.\, u = 0$ in Ω

with the VSB conditions

$v\,.\,n = 0, n \times \nabla \times v = \beta v$ on $\partial\Omega$

$u\,.\,n = 0, n \times \nabla \times u = \beta u$ on $\partial\Omega$

and $\alpha = 0$ satisfies the energy inequality

$$\frac{d}{dt}\|v^0\|^2 + 2a_\beta(v^0,v^0) \le 0$$

5.2 Strong Convergence and the Strong Solutions of the NS:

Theorem 5.2: Let $v_0 \in V$ and (v^α, u^α) be the strong solution stated, let $v_0 \in H$ and $\alpha > 0$. Then there is a time $T^* = T^*(v_0) > 0$ such that the problem has a unique weak solution of (v,u) with initial data v_0 on the interval $[0, T^*)$ in the sense of any $T \,\epsilon\, (0, T^*)$, which satisfies the energy equation

$$\frac{d}{dt} \| v \|^2 + 2\alpha_\beta(v,v) + (B(v.u),v) = 0, on\ [0,T]$$

in the sense of distribution. Furthermore, if $v_0 \in V$ then

$v \,\epsilon\, L^2(0,T;\ W_\beta \cap H) \cap C([0,T);V)$

$$v' \epsilon L^2(0,T;H)$$

and the energy equation

$$\frac{d}{dt}a_\beta(v,v) + 2 \| P\Delta v \|^2 + 2(B(v), -\Delta v) = 0$$

is valid and corresponding to the parameter $\alpha > 0$. Then there is a $T > 0$ and a v^0 in $L^\infty(0,T;V) \cap L^2(0,T;W_\beta \cap H)$ such that

$v^\alpha \longrightarrow v^0$ in $L^2(0,T;W_\beta \cap H)$ weakly

$v^\alpha \longrightarrow v^0$ in $L^2(0,T;V)$ strongly

$u^\alpha \longrightarrow v^0$ in $L^2(0,T;W_\beta \cap H)$ weakly

$u^\alpha \longrightarrow v^0$ in $L^2(0,T;V)$ strongly

with v^0 being a weak solution to the initial boundary problem of the NS equation

$\partial_\tau v - \Delta v + \Delta \times v \times u + \nabla p = 0$ in Ω

$\nabla \,.\, v = 0$ in Ω

$u - \alpha\Delta u + \nabla\tilde{p} = v$ in Ω

$\nabla \,.\, u = 0$ in Ω

with the VSB conditions

$v\,.\,n = 0, n \times \nabla \times v = \beta v$ on $\partial\Omega$

$u\,.\,n = 0, n \times \nabla \times u = \beta u$ on $\partial\Omega$

with $\alpha = 0$ which is unique and thus called the strong solution.

Consequently, it can be extended to the maximal existence time interval $[0,T^*)$ such that if $T^* < \infty$ then $\|v^0\|_1 \longrightarrow \infty$, as $t \longrightarrow T^*$.

Moreover, the following energy holds:

$$\frac{d}{dt}a_\beta(v^0,v^0) + 2\|P\Delta v^0\|^2 - 2(B(v^0,v^0), P\Delta v^0) = 0$$

Proof:

By energy equation

$$v_j'(t) + \lambda_j v_j(t) + g_j(v) = 0$$

$$\frac{d}{dt}a_\beta(v^\alpha,v^\alpha) + \|P\Delta v^\alpha\|^2 \le c\|B(v^\alpha,u^\alpha)\|^2$$

In above equation we get,

$$\|B(v^\alpha,u^\alpha)\|^2 \le c\int_\Omega |\nabla \times v^\alpha|^2 \left|u^\alpha\right|^2 dx$$

$$\le c\|\nabla \times v^\alpha\|^2_{L^3(\Omega)}\|u^\alpha\|^2_{L^6(\Omega)}$$

$$\|\nabla \times v^\alpha\|^2_{L^3(\Omega)} \le c(\|v^\alpha\| + \|P\Delta v^\alpha\|)\|v^\alpha\|_1$$

$\|u^\alpha\|_{L^6(\Omega)} \le c\|u^\alpha\|_1$ and $\|u^\alpha\|_1 \le c\|v^\alpha\|_1$

which follows from the fact that

$$\|u^\alpha\|^2 + \alpha a_\beta(u^\alpha, u^\alpha) = (v^\alpha, u^\alpha)$$

$$a_\beta(u^\alpha, u^\alpha) + \alpha\|P\Delta u^\alpha\|^2 = a_\beta(u^\alpha, v^\alpha)$$

Consequently,

$$\frac{d}{dt} a_\beta(v^\alpha, v^\alpha) + \frac{1}{2}\|P\Delta v^\alpha\|^2 \le c(1 + \|v^\alpha\|_1^2)\|v^\alpha\|_1^4$$

Combining this with similar estimates for

$$|(A_\beta v_m, \phi)| \le |(v_m, \phi)| + |a_\beta(v_m, \phi)|$$

$$\frac{d}{dt}\left(\|v^\alpha\|^2 + a_\beta(v^\alpha, v^\alpha)\right) + \frac{1}{2}(\|v^\alpha\|^2 + \|P\Delta v^\alpha\|^2) \le c(1 + \widetilde{\alpha_\beta}(v^\alpha, v^\alpha))^3$$

Comparing it with the following ordinary differential equation

$$\frac{d}{dt} y = c(1 + y)^3$$

with $y(0) = \widetilde{\alpha_\beta}(v_0, v_0)$ shows that there is a time T such that v^α is uniform bounded in $L^\infty(0,T;V) \cap L^2(0,T;W_\beta \cap H)$

It follows that

$$\|\nabla \times v^\alpha\|_{L^3(\Omega)}^2 \le c(\|v^\alpha\| + \|P\Delta v^\alpha\|)\|v^\alpha\|_1$$

$$\|u^\alpha\|_{L^6(\Omega)} \le c\|u^\alpha\|_1$$

and

$$\|u^\alpha\|_1 \le c\|v^\alpha\|_1$$

which follows from the fact that

$$\|u^\alpha\|^2 + \alpha a_\beta(u^\alpha, u^\alpha) = (v^\alpha, u^\alpha)$$

$$a_\beta(u^\alpha, u^\alpha) + \alpha\|P\Delta u^\alpha\|^2 = a_\beta(u^\alpha, v^\alpha)$$

Consequently,

$$\frac{d}{dt} a_\beta(v^\alpha, v^\alpha) + \frac{1}{2}\|P\Delta v^\alpha\|^2 \le c(1 + \|v^\alpha\|_1^2)\|v^\alpha\|_1^4$$

and

$(v', w) + \alpha_\beta(v, w) + (B(v,u), w) = 0$, a.e. $t \in [0,T)$. $(v^\alpha)'$ is uniform bounded in $L^2(0,T;H)$.

Hence, by using the standard compactness argument, we find a subsequence v^{α_j} of v^α and a v^0 such that

$v^{\alpha_j} \longrightarrow v^0$ in $L^2(0,T;W_\beta \cap H)$ weakly

$v^{\alpha_j} \longrightarrow v^0$ in $L^2(0,T;V)$ strongly

which enables one to pass to the limit to find $v^0 \epsilon\, C([0,T];V) \cap L^2(0,T;W_\beta \cap H)$ such that (v^0, v^0) is a strong solution of the NS equations.

Let v_1^0 and v_2^0 be two strong solutions to the Navier – Stokes equations with same initial data and $w = v_1^0 - v_2^0$. Then

$$\frac{d}{dt}\|w\|^2 + 2a_\beta(w,w) + 2(B(v_1^0, v_1^0) - B_0(v_2^0, v_2^0), w) = 0$$

Note that

$$|(B(v_1^0, v_1^0) - B(v_2^0, v_2^0), w)| \le |(B(w, v_1^0), w)| + |(B(v_2^0, w), w)| \le \widetilde{\alpha_\beta}(w,w) + c(\|v_1^0\|_{L_\infty}(t) + \|\nabla \times v_2^0\|^4)\|w\|^2$$

which, together with the definition of strong solution and Gronwall's inequality, $\|w\| = 0$.

Thus we have obtained the uniqueness of the strong solution to the initial boundary value problem for the Navier – Stokes equations.

By the standard continuation method, the strong solution can be extended to the maximum existent time interval $[0,T^*) \supset [0,T]$, and the energy equation follows from the smoothness of the solution. Consequently, the convergence of the whole sequence of v^α follows.

Finally, we prove the convergence of u^α.

It follows

$u - \alpha\Delta u + \nabla\tilde{p} = v$ in Ω

$\nabla \times u^\alpha - \alpha\Delta(\nabla \times u^\alpha) = \nabla \times v^\alpha$, in Ω

Taking the inner product of above inequality with $-\Delta(\nabla \times u^\alpha)$ and integrating by part, we get

$$\|\Delta u^\alpha\|^2 + \alpha\|(\nabla \times)^3 u^\alpha\| = (\Delta u^\alpha, \Delta v^\alpha) + \int_{\partial\Omega} \Delta u^\alpha . \beta(v^\alpha - u^\alpha)$$

The last term on the right hand side above, we use the fact

$v^\alpha - u^\alpha = n \times ((v^\alpha - u^\alpha) \times n)$ on $\partial\Omega$

and the Stokes formula to get

$$\int_{\partial\Omega} \Delta u^\alpha . (\beta(v^\alpha - u^\alpha)) = \int_{\partial\Omega} \Delta u^\alpha . (n \times (\beta(v^\alpha - u^\alpha) \times n))$$

$$= \int_{\partial\Omega} (n \times \Delta u^\alpha) . (\beta(u^\alpha - v^\alpha) \times n)$$

$$= \int_\Omega (\nabla \times \Delta u^\alpha) . (\beta(u^\alpha - v^\alpha) \times n) - \int_\Omega \Delta u^\alpha . \nabla \times (\beta(u^\alpha - v^\alpha) \times n)$$

where we have extended β and n smoothly to $\bar{\Omega}$. It follows from $u - \alpha\Delta u + \nabla\tilde{p} = v$ in Ω

$$\|v^\alpha - u^\alpha\|^2 = (-\alpha\Delta u^\alpha, v^\alpha - u^\alpha) \le \alpha\|\Delta u^\alpha\|\|v^\alpha - u^\alpha\|$$

which implies $\|v^\alpha - u^\alpha\| \le \alpha\|\Delta u^\alpha\|$

It follows

$\|u^\alpha\|_1 \le c\|v^\alpha\|_1$ and

$$\|v^\alpha - u^\alpha\| \le \alpha\|\Delta u^\alpha\|$$

$$\left|\int_\Omega \Delta u^\alpha . \nabla \times (\beta(u^\alpha - v^\alpha) \times n)\right| \le \frac{1}{4}\|\Delta u^\alpha\|^2 + c\|v^\alpha\|_1^2$$

for suitably small α.

Using $\|v^\alpha - u^\alpha\| \le \alpha\|\Delta u^\alpha\|$ again gives

$$\left|\int_\Omega (\nabla \times \Delta u^\alpha) . (\beta(u^\alpha - v^\alpha) \times n)\right| \le \frac{1}{2}\alpha \int_\Omega |\nabla \times (\Delta u^\alpha)|^2 + \alpha^{-1} c\|u^\alpha - v^\alpha\|^2 \le \frac{1}{2}\alpha \int_\Omega |\nabla \times (\Delta u^\alpha)|^2 + c\alpha\|\Delta u^\alpha\|^2$$

Collecting

$$\|\Delta u^\alpha\|^2 + \alpha\|(\nabla \times)^3 u^\alpha\| = (\Delta u^\alpha, \Delta v^\alpha) + \int_{\partial\Omega} \Delta u^\alpha . \beta(v^\alpha - u^\alpha)$$

$$= \int_{\partial\Omega} (n \times \Delta u^\alpha) . (\beta(u^\alpha - v^\alpha) \times n)$$

$$\left|\int_\Omega \Delta u^\alpha . \nabla \times (\beta(u^\alpha - v^\alpha) \times n)\right| \le \frac{1}{4}\|\Delta u^\alpha\|^2 + c\|v^\alpha\|_1^2$$

$$\le \frac{1}{2}\alpha \int_\Omega |\nabla \times (\Delta u^\alpha)|^2 + \alpha^{-1} c\|u^\alpha - v^\alpha\|^2$$

i.e., $\|\Delta u^\alpha\|^2 + \alpha\|(\nabla \times)^3 u^\alpha\|^2 \le c\|v^\alpha\|_2^2$

for suitably small α. This, together with the bound of $\partial_t u^\alpha$ in H, implies the desired convergence in

$u^\alpha \longrightarrow v^0$ in $L^2(0,T;W_\beta \cap H)$ weakly

$u^\alpha \longrightarrow v^0$ in $L^2(0,T;V)$ strongly

5.3 Estimates on Convergence Rates:

Finally, we study the rates of convergence in the case of strong solutions. We start with the case that the limiting Navier – Stokes system has a strong solution.

Preposition 5.3: Let $v_0 \in V$ and v^0 be the strong solution to the Navier – Stokes equation with initial data v_0 on any given finite interval $[0,T]$ with $T > 0$. Then there exists a $\alpha_0 > 0$ such that for each $\alpha \in (0,\alpha_0]$, the LNS – α with the initial data v_0 has a unique strong solution (v^α, u^α) on the same interval $[0,T]$ satisfying

$$\sup_{0\le t\le T}\|(v^\alpha,u^\alpha)-(v^0,v^0)\|^2 + \int_0^T \|(v^\alpha,u^\alpha)-(v^0,v^0)\|_1^2(t)dt \le c\alpha$$

$$\sup_{0\le t\le T}\|v^\alpha - v^0\|_1^2 + \int_0^T \|v^\alpha - v^0\|_2^2 dt \le c\alpha^{\frac{1}{2}}$$

with c being a positive constant depending on v^0.

RESULT

It is not clear to us whether the stronger estimate as

$$\sup_{0\le t\le T}\|v^\alpha - v^0\|_1^2 + \int_0^T \|v^\alpha - v^0\|_2^2 dt \le c\alpha^{\frac{1}{2}}$$

with c being a positive constant depending on v^0 holds for u^α under the assumptions in below:

Let $v_0 \in V$ and v^0 be the strong solution to the Navier – Stokes equation with initial data v_0 on any given finite interval $[0,T]$ with $T > 0$. Then there exists a $\alpha_0 > 0$ such that for each $\alpha \in (0,\alpha_0]$, the LNS – α with the initial data v_0 has a unique strong solution (v^α, u^α) on the same interval $[0,T]$ satisfying

$$\sup_{0\le t\le T}\|(v^\alpha,u^\alpha)-(v^0,v^0)\|^2 + \int_0^T \|(v^\alpha,u^\alpha)-(v^0,v^0)\|_1^2(t)dt \le c\alpha$$

$$\sup_{0\le t\le T}\|v^\alpha - v^0\|_1^2 + \int_0^T \|v^\alpha - v^0\|_2^2 dt \le c\alpha^{\frac{1}{2}}$$

with c being a positive constant depending on v^0.

However, under the additional assumption that the strong solution

$v^0 \in L^\infty([0,T],H^2)$, there holds also

$$\sup_{0\le t\le T}\|u^\alpha - v^0\|_1^2 + \int_0^T \|u^\alpha - v^0\|_2^2 dt \le c\alpha^{\frac{1}{2}}$$

This follows from

$$a_\beta(u^\alpha - v^0, u^\alpha - v^0) + \alpha\big(P\Delta(u^\alpha - v^0),\ P\Delta(u^\alpha - v^0)\big)$$

$$= a_\beta(v^\alpha - v^0, u^\alpha - v^0) + \alpha\big(-P\Delta v^0, P\Delta(u^\alpha - v^0)\big)$$

$$a_\beta(u^\alpha - v^0, u^\alpha - v^0) + \alpha\big(P(\Delta u^\alpha), \Delta(u^\alpha - v^0)\big)$$

$$= a_\beta(v^\alpha - v^0, u^\alpha - v^0)$$

and

$$\sup_{0\le t\le T}\|v^\alpha - v^0\|_1^2 + \int_0^T \|v^\alpha - v^0\|_2^2 dt \le c\alpha^{\frac{1}{2}}$$

CONCLUSION

The energy equation which yields the global existence directly, and the corresponding convergence result is better both in v^α and u^α.

REFERENCES

[1] Constantin P and Foias C (1988) Navier Stokes Equation. University of Chicago Press, Chicago.

[2] Cantarella J, De Turk D and Gluck H(2002) Vector calculus and the topology of domains in 3 – space. *Am. Math. Mon*. 109: 257.

[3] Bellout H, Neustupa J and Penel P (2004) On the Navier – Stokes equations with boundary conditions based on vorticity. *Math. Nachr*. 269: 270, 59C72.

[4] Beirao da Veiga H and Crispo F (2010) Sharp inviscid limit results under Navier type boundary conditions. An Lp theory. *J. Math. Fluid Mech*. 12: 397 – 411.

[5] Beirao da Veiga H and Crispo, F (2011) Concerning the $W^{k,p}$ inviscid limit for 3D flows under a slip boundary condition. *J. Math. Fluid Mech*. 13: 117 – 135.

[6] Constantin P and Iyer G (2008) A stochastic Lagrangian representation of the 3D incompressible Navier - Stokes equations. *Pure Appl. Math*. 61(3): 330–345.

Citation : Kavitha S and Jeevitha K (2014) Vorticity and Navier-Slip boundary conditions in 3D Lagrangian Navier- Stokes α model, *Int J Adv Interdis Res*, 1(9): 1-7.

Received : August 21,2014 | **Revised :** August 23,2014 | **Accepted :** August 24,2014 | **Published:** August 26,2014