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On 3D Lagrangian Navier-Sokes α Model with a class of Vorticity-Slip Boundary Conditions

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Abstract

This paper concerns the 3-dimensional Lagrangian Navier-Stokes α model and the limiting Navier-Stokes system on smooth bounded domains with a class of vorticity-slip boundary conditions and the Navier-slip boundary conditions. It establishes the spectrum properties and regularity estimates of the associated Stokes operators, the local well-posedness of the strong solution and global existence of weak solutions for initial boundary value problems for such systems. Furthermore, the vanishing α limit to a weak solution of the corresponding initial-boundary.

Keywords Navier-Stokes α model, vorticity-slip boundary conditions, Vanishing α limit.

INTRODUCTION

The Lagragian Navier-Stokes α model (LNS-α) as a regularization system of the NavierStokes equations (NS) is given by

$$\partial_t v - \Delta v + T_\alpha v.\nabla v + \nabla(T_\alpha v)^T \cdot v = 0 \quad \rightarrow (1.1)$$

$$\nabla.u = 0 \quad \rightarrow (1.2)$$

which describes large scale fluid motions in the turbulence theory, where$T_\alpha v = u$ is a filtered version of the velocity v determined usually

$$u - \alpha\Delta u = v \quad \rightarrow (1.3)$$

$$\nabla.u = 0 \quad \rightarrow (1.4)$$

bywith $\alpha > 0$ being a constant. This filter u is also called the averaged velocity. The system can be regarded as a system for this filter, and is also called the Lagrangian averaged Navier-Stokes equations (LANS). The global well-posedness for the LANS was first obtained in [1] for periodic boundary conditions. The convergence of its solutions to that of the NS equations and the continuity of attractors when $\alpha \rightarrow 0$ are also considered there.

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when $\alpha \rightarrow 0$ are also considered there.

For bounded domains, the situation becomes more complicated since the LANS is a 4th odder system for the filter u, and only the no-slip boundary condition $u = 0$ on the boundary was considered by [2] under the assumption that $Au = -P\Delta u = 0$ on the boundary with P being the Leray projection operator. The boundary effects related to such a boundary condition were analyzed in [3].

On the other hand, the LNS-α model emphasizes the system (1.1)-(1.4) as equations for the physical velocity v, which is a regularized system of the NS equations by filtering some part of the nonlinearity through a global quantity which is then called filtered velocity .It is also mentioned in [18] in the stochastic Lagrangian derivation of (1.1), (1.2) that any translation-invariant filter $u = T_\alpha v$ may be adaptable.Although, there is no any serious difference between the two aspects for the equations (1.1), (1.2) filtered by (1.3), (1.4) in domains without boundary, the situation may be different for domains with boundaries. To our knowledge, very little is known to the LNS-α models in domains with boundaries from this point of view.

In this paper, we investigate the initial boundary value problem for the LNS-α model

(1.1), (1.2) in the following equivalent form

$$\partial_t v - \Delta v + T_\alpha v . \nabla v + \nabla (T_\alpha v)^T \cdot v = 0 \quad \rightarrow (1.5)$$

$$\nabla . u = 0 \quad \rightarrow (1.6)$$

2.On 3D Lagrangian Navier – Stokes α Model with a class of Vorticity-Slip Boundary Conditions

2.1 Theorem

The stokes operator $A_F = -\Delta$ with the domain $D(A_F) = W \cap FH$ is self-adjoint in the Hilbert space FH.

Proof:

Given:

The stokes operator $A_F = -\Delta$ with the domain

$D(A_F) = W \cap FH$

To prove:

Self-adjoint in the Hilbert space FH.

It is clear that $A_F = -\Delta$ with the domain $W \cap FH$ is symmetric.

Since $C_0^\infty(\Omega) \cap H$ is dense in H,

It follows that A_F is densely defined due to the orthogonality of FH and HH and the compactness of HH.

Let $u \in W$. Since $n \times (\nabla \times u) = 0$ on $\partial\Omega$, then $-\nabla u = \nabla \times (\nabla \times u) \in FH$.

Thus A_F maps $W \cap FH$ to FH.

Now, for any $f \in FH$, it follows that there is a $\varphi \in H^1(\Omega)$ satisfying

$$\nabla \times \varphi = f \quad \text{in } \Omega$$

$$\nabla . \varphi = 0 \text{ in } \Omega$$

$$\varphi \times \mathrm{n} = 0 \text{ on } \partial\Omega$$

There is a $v \in FH \cap H^2(\Omega)$ so that

$$\varphi = \nabla \times v + P_{HG}^{\varphi}$$

Here $P_{HG}^{\varphi} \times n = 0$ on $\partial\Omega$.

It follws that $n \times (\nabla \times v) = 0$ on $\partial\Omega$

Then $\nabla \times (P_{HG}^{\varphi}) = 0$ and $\varphi = \nabla \times v + P_{HG}^{\varphi}$

implies that $-\nabla \mathrm{v} = f$ in Ω

Thus

$A_F: W \cap FH \rightarrow FH$ is surjective. If $f = 0$, then integration by parts shows

$$\|\nabla \times v\| = 0$$

It follows that $\mathrm{u} = 0$ due to the orthogonality of FH and HH and

Then $A_F: W \cap FH \rightarrow FH$ is one to one.

Here W and FH are closed in $H^2(\Omega)$ and $L^2(\Omega)$, and

$$\|\Delta v\| \leq \|v\|_2$$

We obtain from the Banach inverse operator theorem that

$$\|v\|_2 \leq c\|\nabla v\|$$

Hence the proof.

2.2 Theorem

The operator A_F and the stokes operator $A_F = -\Delta$ with the domain $DA_F = W \cap FH$ is self-adjoint in the Hilbert space FH is the self adjoint extension of the following bilinear form

$$a(u, \varphi) = (\nabla \times u, \nabla \times \varphi), \mathrm{D(a)}$$

$$D(a) = V_F = FH \cap H^1(\Omega) \text{ in } FH.$$

Proof:

Given:

The operator A_F and the stokes operator $A_F = -\Delta$ with the domain $DA_F = W \cap FH$ is self-ad joint in the Hilbert space FH is the self adjoint.

To prove :

The self adjoint extension of the following bilinear form

$$a(u, \varphi) = (\nabla \times u, \nabla \times \varphi), \mathrm{D(a)}$$

$$D(a) = V_F = FH \cap H^1(\Omega) \text{ in } FH.$$

$a(u, \varphi)$ with $D(a) = FH \cap H^1(\Omega)$ is densely defined.

$a(u, \varphi)$ is closed and positive. It follows that there is a self-adjoint operator

A with domain $D(A) \subset D(a)$ Such that

$$a(u,\varphi) = (Au,\varphi), \forall \varphi \in FH \cap H^1(\Omega)$$

For any $u \in D(A)$. It is clear that

$$D(A_F) = W \cap FH \subset D(A)$$

And $Au = -\Delta u$ for any $u \in W \cap FH$.

let $u \in D(A)$ and $f = Au$. It follows from (Theorem 1) That there is a $v \in D(A_F)$ such that

$$a(v,\varphi) = (f,\varphi)$$

$\forall \varphi \in V_F$.

On the other hand

$$a(u,\varphi) = (Au,\varphi) = (f,\varphi)$$

$\forall \varphi \in V_F$,

Hence $a(u-v,\varphi) = (\nabla \times (u-v), \nabla \times \varphi) = 0 \forall \varphi \in V_F$,

Taking $\varphi = u - v$ shows that $\nabla \times (u-v) = 0$.

Thus $u = v$.

Thus $D(A) = D(A_F)$ and $A = A_F$.

Denote by V_F^1 the dual of V_F respect to the L^2 inner product.

Then the notation of weak solutions can be extended for $f \in V_F^1$: u is called a weak solution to $f \in V_F^1$ if

$$a(u,\varphi) = (f,\varphi),$$

$\forall \varphi \in V_F$

Hence the proof.

2.3 Theorem

The self-adjoint extension of the bilinear form $\tilde{a}_\gamma(u,\varphi)$ with domain $D(\tilde{a}_\gamma) = V$ is the stokes operator $A_\gamma = I + P(-\Delta)$ with $D(A_\gamma) = \widetilde{W}_\gamma \cap H$, and A_γ is an isomorphism between $D(A_\gamma)$ and H with a compact inverse on H. Consequently,the eigenvalues of the stokes operator A_γ can be listed as

$$1 \le 1 + \lambda_1 \le 1 + \lambda_2 \dots \dots \to \infty$$

With the corresponding eigenvectors $\{e_j\} \subset \widetilde{W}_\gamma, i.e.,$

$$A_\gamma e_j = (1 + \lambda_j) e_j$$

Which form a complete orthogonal basis in H.Furthermore, it holds that

$$(1 + \lambda_1)\gamma \le \tilde{a}_\beta(u,u) \le \frac{1}{1+\lambda_1} \|A_\gamma u\|^2, \forall u \in D(A_\gamma)$$

Proof:

Given:

The self-adjoint extension of the bilinear form $\tilde{a}_\gamma(u,\varphi)$ with domain $D(\tilde{a}_\gamma) = V$ is the stokes operator $A_\gamma = I + P(-\Delta)$ with $D(A_\gamma) = \widetilde{W}_\gamma \cap H$, and A_γ is an isomorphism between $D(A_\gamma)$ and H with a compact inverse on H

To prove:

The eigenvalues of the stokes operator A_γ can be listed as

$$1 \le 1 + \lambda_1 \le 1 + \lambda_2 \dots \dots \to \infty$$

With the corresponding eigenvectors $\{e_j\} \subset \widetilde{W}_\gamma, i.e.,$

$$A_\gamma e_j = (1 + \lambda_j) e_j$$

Which form a complete orthogonal basis in H.Furthermore, it holds that

$$(1 + \lambda_1)\|u\|^2 \le \tilde{a}_\beta(u,u) \le \frac{1}{1+\lambda_1} \|A_\gamma u\|^2, \forall u \in D(A_\gamma)$$

It suffices to show that $D(A_\gamma) \subset \widetilde{W}_\gamma \cap H$

Let $u \in D(A_\gamma)$and $f = A_\gamma u$.

Since $D(A_\gamma) \subset D(\tilde{a}_\beta) = H^1(\Omega) \cap H$,

It follows that $\tilde{a}_\gamma(u,\varphi) = (f,\varphi), \forall \varphi$ in V that

$$\|u\|_1^2 \le c\|f\|^2$$

Let $n(x)$ and $\gamma(x)$ be internal smooth extensions of the normal vector n and γ.

Then $(\gamma(x)u + GD(u)) \times n(x) = \nabla \times v + \nabla h + \nabla g$

With $v \in H^2(\Omega) \cap FH$,

$$\nabla h = P_{HG}((\gamma(x)u + GD(u)) \times n(x))$$

and

$$\nabla_g = P_{GG}((\gamma(x)u + GD(u)) \times n(x))$$

One can get

$$\|v\|_2 \le c\|\nabla\times v\|_1 \le c\|u\|_1$$

Here $n\times(\nabla h)=0$ and $n\times(\nabla g)=0.$

Thus $(\gamma u+GD(u)).\varphi=(-\Delta v,\varphi),\forall\varphi\in V$

$$\int_\Omega(\nabla\times v).(\nabla\times\varphi)$$

$$+\int_{\partial\Omega}(\gamma u+GD(u)).\varphi=(-\Delta v,\varphi),\forall\varphi\in V$$

Then the definition of the weak solution

$$\|\Delta v\| \le \|v\|_2$$

$$\int_\Omega(\nabla\times v).(\nabla\times\varphi)$$

$$+\int_{\partial\Omega}\gamma u.\varphi+\int_{\partial\Omega}GD(\varphi).u=(f-u,\varphi),\forall\in V$$

Combine the equation

$$\int_{\partial\Omega}GD(\varphi).u=\int_{\partial\Omega}GD(u).\varphi$$

$$\int_\Omega(\nabla\times(u-v)).(\nabla\times\varphi)$$

$$=(P_{FH}(f-u+\Delta v),\varphi),\forall\varphi\in V$$

Here $\nabla\times u=\nabla\times P_{FH}(u)$ and $P_F(u)\in H^1(\Omega)\cap FH$. It follows that

a($P_{FH}(u\quad)\quad -v,\varphi)=(P_{FH}(f-u+\Delta v),\varphi),\forall\varphi\in H^1(\Omega)\cap FH$

since $P_{FH}(f-u+\Delta v)\in FH$, so $P_{FH}(u)-v\in W$,

$$\|P_{FH}(u)-v\|_2\le c(\|f\|+\|u\|_1)$$

since HH is finite dimensional,

so $\|P_{HH}(u)\|_2\le c\|u\|$

It follows from the equation

$$\|u\|_1^2\le c\|f\|^2$$

$$\|v\|_2\le c\|\nabla\times v\|_1\le c\|u\|_1$$

$$\|P_{FH}(u)-v\|_2\le c(\|f\|+\|u\|_1)$$

$$\|P_{HH}(u)\|_2\le c\|u\|$$

$$\|u\|_2\le c\|f\|$$

$$(\nabla\times u)\times n=(\nabla\times P_{FH}(u))\times n=(\nabla\times u)\times n$$

$$=-\gamma u-GD(u)$$

$$2(S(u)n)_\tau$$

$$=((\nabla\times u)\times n+GD(u))_\tau$$

$$2(S(u)n)_\tau=-\gamma u_\tau$$

Hence the proof.

3. Vanishing α limit and the Navier Stokes equations

3.1 Theorem

Let $v_0\ \in H$, and (v^α,u^α) be the global weak solution if $v_0\ \in H,\alpha>0$, then the solution v obtained is global

$$T^*=T^*(v_0)=\infty$$

corresponding to the parameter $\alpha>0$.Then for any given $T>0$ there is a subsequence $u^{\alpha j}$ of u^α and $a(v^0,u^0)$ statisfying

$$v^0\in L^2(0,T;V)\cap C_w([0,T];H)$$

$$(v^0)'\in L^{\frac{4}{3}}(0,T;V')$$

$$v^{\alpha j}\to v^0 \text{ in } L^2(0,T;H) \text{ weakly}$$

$$v^{\alpha j}\to v^0 \text{ in } L^2(0,T;D\left(A_\beta^{\frac{-1}{4}}\right)) \text{ strongly}$$

$$u^{\alpha j}\to v^0 \text{ in } L^2(0,T;V_\beta) \text{ weakly}$$

$$u^{\alpha j}\to v^0 \text{ in } L^2(0,T;D\left(A_\beta^{\frac{-1}{4}}\right)) \text{ strongly}$$

Moreover (v^0,u^0) is a weak solution of the intial boundary problem of the NS equations with $\alpha=0$ and satisfies the energy inequality

$$\frac{d}{dt}\|v^0\|^2+2a_\beta(v^0,v^0)\le 0$$

Proof:

Given:

Let $v_0\ \in H$, and (v^α,u^α) be the global weak solution if $v_0\ \in H,\alpha>0,$

To prove:

then the solution v obtained is global

$$T^*=T^*(v_0)=\infty$$

corresponding to the parameter $\alpha>0$.Then for any given $T>0$ there is a subsequence $u^{\alpha j}$ of u^α and $a(v^0,u^0)$ statisfying

$$v^0\in L^2(0,T;V)\cap C_w([0,T];H)$$

$$(v^0)'\in L^{\frac{4}{3}}(0,T;V')$$

A weak solution of the intial boundary problem of the NS equations with $\alpha=0$ and satisfies the energy inequality

$$\frac{d}{dt}\|v^0\|^2 + 2a_\beta(v^0, v^0) \le 0$$

Let $v_0 \in H$, $T > 0$, and (v^α, u^α) be the global weak solution corresponding to $1 \ge \alpha > 0$. It follows that

$$\frac{d}{dt}\left(\|u\|^2 + \alpha a_\beta(u,u)\right) + 2\left(a_\beta(u,u) + \alpha\|P\Delta u\|^2\right) = 0$$

$$+a a_\beta(u^\alpha, u^\alpha)$$

$$+\int_0^t \left(a_\beta(u^\alpha, u^\alpha) + \alpha\|P\Delta u^\alpha\|^2\right)dt \le c$$

For some constant c independent of α. For any $\varphi \in W_\beta \cap H$, we have

$$(B(v^\alpha, u^\alpha), \varphi) = \int_\Omega (\nabla \times v^\alpha \times u^\alpha)\varphi dx = I + II$$

Where $I = \int_{\partial\Omega}(n \times v^\alpha).(v^\alpha \times \varphi)dS$

$II = \int_\Omega v^\alpha.(-u^\alpha.\nabla\varphi - \varphi.\nabla u^\alpha)dx$

Since $u.n = 0$ and $\varphi.n = 0$ on the boundary so

$$u^\alpha \times \varphi = \lambda n \text{ on } \partial\Omega$$

Hence $I = 0$

To estimate

$$II = \int_\Omega v^\alpha.(-u^\alpha.\nabla\varphi - \varphi.\nabla u^\alpha)dx$$

Here

$$\left|\int_\Omega v^\alpha(v^\alpha.\nabla\varphi)dx\right| \le c(\|u^\alpha\| + \alpha\|P\Delta u^\alpha\|)\|u^\alpha\|_{L^3(\Omega)}\|\nabla\varphi\|_{L^6(\Omega)}$$

$$\|u^\alpha\|^2_{L^3(\Omega)} \le c\|u^\alpha\|\|u^\alpha\|_1$$

$$\le c\|u^\alpha\|^{\frac{3}{2}}(\|u^\alpha\| + \|P\Delta u^\alpha\|)^{\frac{1}{2}}$$

$$\|\nabla\varphi\|_{L^6(\Omega)} \le c\|A_\beta\varphi\|$$

Then

$$\|u^\alpha\|^2 + a a_\beta(u^\alpha, u^\alpha)$$

$$+\int_0^t \left(a_\beta(u^\alpha, u^\alpha) + \alpha\|P\Delta u^\alpha\|^2\right)dt \le c$$

And

$$\left|\int_\Omega v^\alpha(u^\alpha.\nabla\varphi)dx\right| \le c(\left(a_\beta(u^\alpha, u^\alpha)^{\frac{1}{2}} + \alpha\|P\Delta u^\alpha\| + \alpha\|P\Delta u^\alpha\|)^{\frac{5}{4}}\right)\|A_\beta\varphi\|$$

Next, $\left|\int_\Omega v^\alpha(\varphi.\nabla u^\alpha)dx\right|$

$$\le c\,(\|u^\alpha\| + \alpha\|P\Delta u^\alpha\|)\|u^\alpha\|_1\,\|\varphi\|_{L^\infty(\Omega)}$$

Which implies that

$$\left|\int_\Omega v^\alpha(\varphi.\nabla u^\alpha)dx\right| \le c(\left(a_\beta(u^\alpha, u^\alpha)^{\frac{1}{2}} + \alpha\|P\Delta u^\alpha\| + \alpha\|P\Delta u^\alpha\|)^{\frac{3}{2}}\right)\|A_\beta\varphi\|$$

Then for $\alpha < 1$, $|(B(v^\alpha, u^\alpha), \varphi)| \le c(1 + (a_\beta(u^\alpha, u^\alpha)^{\frac{1}{2}} + \alpha^{\frac{3}{4}}\|P\Delta u^\alpha\|^{\frac{3}{2}})\,\|A_\beta\varphi\|$

It follows from the equation

$$\|u^\alpha\|^2 + a a_\beta(u^\alpha, u^\alpha)$$

$$+\int_0^t \left(a_\beta(u^\alpha, u^\alpha) + \alpha\|P\Delta u^\alpha\|^2\right)dt \le c$$

And $|(B(v^\alpha, u^\alpha), \varphi)| \le c(1 + (a_\beta(u^\alpha, u^\alpha)^{\frac{1}{2}} + \alpha^{\frac{3}{4}}\|P\Delta u^\alpha\|^{\frac{3}{2}})\,\|A_\beta\varphi\|$

$B(v^\alpha, u^\alpha)$ and then $\frac{d}{dt}(v^\alpha)$ are uniformly bounded in $L^{\frac{4}{3}}(0,T; D(A_\beta^{-1}))$.

It follows that

$$(1-\alpha)u_t^\alpha + \alpha A_\beta(u_t^\alpha) = v_t^\alpha$$

$$(1-\alpha)\left\|A_\beta^{-1}u_t^\alpha\right\|^2 + \alpha\left\|A_\beta^{-1}u_t^\alpha\right\|^2 = \left\|A_\beta^{-1}u_t^\alpha\right\|^2$$

$$\left\|A_\beta^{-1}u_t^\alpha\right\|^2 \le 2\left\|A_\beta^{-1}u_t^\alpha\right\|^2$$

For $0 < \alpha \le \frac{1}{2.}$ This shows that $\partial_t u^\alpha$ are uniformly bounded in $L^{\frac{4}{3}}(0,T; D(A_\beta^{-1}))$ as $\partial_t u^\alpha$.

It implies that (u^α) are uniformly bounded in $L^2(0,T;V)$.

And the duality between $V = D(A_\beta^{\frac{1}{2}})$ and $D(A_\beta^{-1})$ with respect to the inner product of $D(A_\beta^{\frac{-1}{4}})$

$$\left(A_\beta^{\frac{-1}{4}}u, A_\beta^{\frac{1}{2}}\varphi\right) = \left(A_\beta^{\frac{1}{2}}u, A_\beta^{-1}\varphi\right)$$

There exist a subsequence

$u^{\alpha j}$ of u^α and v^0 such that

$u^{\alpha j} \to v^0$ in $L^2(0,T;V)$ weakly

$u^{\alpha j} \to v^0$ in $L^2\left(0,T;D\left(A_\beta^{\frac{-1}{4}}\right)\right)$ strongly

Here

$$|(B(v^\alpha,u^\alpha) - B(v^0,v^0),\varphi)| \le I + II$$

Where

$$I = |(B(u^\alpha - v^0,u^\alpha) + B(v^0,u^\alpha - v^0),\varphi)|$$

$$II = \alpha|(B(P\Delta u^\alpha,u^\alpha),\varphi)|$$

Similar to the equation

$$(B(v^\alpha,u^\alpha),\varphi) = \int_\Omega (\nabla\times v^\alpha\times u^\alpha)\varphi dx = I + II$$

And $I = 0$

Integrating by part we get,

$|(B(u^\alpha - v^0,u^\alpha),\varphi)| = \left|\int_\Omega (u^\alpha - v^0).(u^\alpha.\nabla\varphi + \varphi.\nabla u^\alpha)\right|$

$$\left|\int_\Omega (u^\alpha - v^0).(u^\alpha.\nabla\varphi)\right|$$

$$\le c\|u^\alpha - v^0\|\|u^\alpha\|^{\frac{1}{2}}\|u^\alpha\|^{\frac{1}{2}}_{L^6(\Omega)}\|\nabla\varphi\|_{L^6(\Omega)}$$

And

$$\|u^\alpha - v^0\|^2 \le c\|u^\alpha - v^0\|_{D(A_\beta^{\frac{-1}{4}})}\|u^\alpha - v^0\|_{D(A_\beta^{\frac{1}{4}})}$$

$$\|u^\alpha - v^0\|^2_{D(A_\beta^{\frac{1}{4}})} \le c\|u^\alpha - v^0\|\|u^\alpha - v^0\|_1$$

This shows that

$$\left|\int_\Omega (u^\alpha - v^0)\right|.(u^\alpha.\nabla\varphi)$$

$$\le \|u^\alpha - v^0\|^{\frac{1}{2}}_{D(A_\beta^{\frac{-1}{4}})}\|u^\alpha - v^0\|_1^{\frac{1}{4}}\|u^\alpha\|_1^{\frac{1}{2}}\|A_\beta\varphi\|$$

Hence

$$\left|\int_\Omega (u^\alpha - v^0)\right|.(u^\alpha.\nabla\varphi)$$

$$\le \|u^\alpha - v^0\|^{\frac{1}{2}}_{D\left(A_\beta^{\frac{-1}{4}}\right)}(\|u^\alpha\|_1^{\frac{3}{4}} + \|v^0\|_1^{\frac{3}{4}}\|A_\beta\varphi\|$$

$$\left|\int_\Omega (v^0 - v^0).(\varphi.\nabla u^\alpha)\right|$$

$$\le \|u^\alpha - v^0\|\|u^\alpha\|_1\|\varphi\|_{L^\infty(\Omega)}$$

It follows that

$$\left|\int_\Omega (u^\alpha - v^0)\right|.(u^\alpha.\nabla\varphi)$$

$$\le \|u^\alpha - v^0\|^{\frac{1}{2}}_{D\left(A_\beta^{\frac{-1}{4}}\right)}(\|u^\alpha\|_1^{\frac{5}{4}} + \|v^0\|_1^{\frac{5}{4}}\|A_\beta\varphi\|$$

Then

$$|(B(v^o - u^\alpha,v^o),\varphi)| \le c\|u^\alpha - v^0\|^{\frac{1}{2}}_{D\left(A_\beta^{\frac{-1}{4}}\right)}$$

$$(1 + \|u^\alpha\|_1^{\frac{5}{4}} + \|v^0\|_1^{\frac{5}{4}}\|A_\beta\varphi\|$$

Similarly,one can obtain

$$|(B(u^\alpha - v^0,u^\alpha),\varphi)|$$

$$\le \|u^\alpha - v^0\|^{\frac{1}{2}}_{D\left(A_\beta^{\frac{-1}{4}}\right)}(1 + \|u^\alpha\|_1^{\frac{5}{4}} + \|v^0\|_1^{\frac{5}{4}}\|A_\beta\varphi\|$$

It follows that

$$I \le \|u^\alpha - v^0\|^{\frac{1}{2}}_{D\left(A_\beta^{\frac{-1}{4}}\right)}(1 + \|u^\alpha\|_1^{\frac{5}{4}} + \|v^0\|_1^{\frac{5}{4}}\|A_\beta\varphi\|$$

Similarly,

$$|(B(P\Delta u^\alpha,u^\alpha),\varphi)| = \left|\int_\Omega (P\Delta u^\alpha).(u^\alpha.\nabla\varphi + \varphi.\nabla u^\alpha)\right|$$

Then

$$|(B(P\Delta u^\alpha,u^\alpha),\varphi)| \le c\|P\Delta u^\alpha\|\|u^\alpha\|_1\|A_\beta\varphi\|$$

It follows that

$$II \le c\alpha^{\frac{1}{2}}(\alpha\|P\Delta u^\alpha\|^2 + \|u^\alpha\|_1^2)\ \|\varphi\|_2$$

It follows that the equation

$$(B(v^\alpha, u^\alpha), \varphi) = \int_\Omega (\nabla \times v^\alpha \times u^\alpha)\varphi dx = I + II$$

$$I \leq \|u^\alpha - v^0\|^{\frac{1}{2}}_{D\left(A_\beta^{\frac{-1}{4}}\right)} (1 + \|u^\alpha\|_1^{\frac{5}{4}} + \|v^0\|_1^{\frac{5}{4}} \|A_\beta \varphi\|$$

$$II \leq c\alpha^{\frac{1}{2}}(\alpha\|P\Delta u^\alpha\|^2 + \|u^\alpha\|_1^2)\ \|\varphi\|_2$$

This equation becomes

$B(v^{\alpha j}, u^{\alpha j}) \rightarrow B(v^0, v^0)$ in $L^1(0,T; D(A_\beta^{-1}))$ strongly

To show that v^0 satisfies

$$((v^0)', \varphi) + a_\beta(v^0, \varphi) + \left((\nabla \times v^0) \times v^0, \varphi\right) = 0$$

$\forall\ \varphi \in C^\infty(\Omega) \cap V$ in the sense of distribution on $[0, T]$.

Here $v^0 \in L^2(0,T; V)$ implies $(v^0)' \in L^{\frac{4}{3}}(0,T; V')$

Thus $\forall \varphi \in V$.

$$\frac{d}{dt}\left(\|u^\alpha\|^2 + \alpha a_\beta(u^\alpha, u^\alpha)\right) + 2a_\beta(u^\alpha, u^\alpha) \leq 0$$

Passing to the limit and nothing the weak lower semi-continuity of the norm, one gets

$$\frac{d}{dt}\|v^0\|^2 + 2a_\beta(v^0, v^0) \leq 0$$

Here

$$(v^\alpha, v^0, \varphi) = (u^\alpha - v^0, \varphi) + \alpha(\left(A_\beta^{\frac{3}{4}} u^\alpha, A_\beta^{\frac{1}{4}} \varphi\right) - (u^\alpha, \varphi))$$

For $\varphi \in D\left(A_\beta^{\frac{-1}{4}}\right)$.Then

$$\|v^\alpha - v^0\|^2_{D(A_\beta^{\frac{-1}{4}})} \leq \|v^\alpha - v^0\|^2_{D(A_\beta^{\frac{-1}{4}})} + c\alpha^{\frac{1}{2}}(\alpha\|P\Delta u^\alpha\|^2 + \|u^\alpha\|_1^2$$

It follows that

$$v^{\alpha j} \rightarrow v^0 \text{ in } L^1\left(0,T; D\left(A_\beta^{\frac{-1}{4}}\right)\right)$$

strongly

Here

$$(v^\alpha - v^0, \varphi) = (u^\alpha, v^0, \varphi) - \alpha(P\Delta u^\alpha, \varphi)$$

It follows from the equation

$$\|u^\alpha\|^2 + a a_\beta(u^\alpha, u^\alpha) + \int_0^t \left(a_\beta(u^\alpha, u^\alpha) + \alpha\|P\Delta u^\alpha\|^2\right)dt \leq c$$

And

$u^{\alpha j} \rightarrow v^0$ in $L^2(0,T; V)$ weakly and we get

$v^{\alpha j} \rightarrow v^0$ in $L^1(0,T; H)$ weakly

Hence the proof.

CONCLUSION

Throughout this paper we discussed some definition and theorems on $3D$ lagrangian Navier-stokes α model with a class of vorticity –slip boundary condition. And then we discussed a vanishing α limit and the Navier stokes equation . The vanishing α limit to a weak solution of the corresponding initial-boundary value problem of the Navier-Stokes system is proved and a rate of convergence is shown for the strong solution.

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