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Stokes and Navier-Stokes Equation with a Non-homogeneous Divergence Condition

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Abstract

In this paper, we study the existence and regularity of solution to the stokes and oseen equation with a non-homogeneous divergence condition. we also prove the existence of global weak solution to the 3D Navier-stokes equations when the divergence is not equal to zero. These equations intervene in control problems for the Navier- stokes equations and in fluid –structure interaction problems.

Keywords Navier-Stokes equation , nonhomogeneous divergence condition ,.non-homogeneous Dirichlet boundary condition ..

INTRODUCTION

Let Ω be a bounded and connected domain in $\mathbb{R}^N$, with N = 2 or N = 3, with a regular boundary Γ ,and let T be positive .

Set $Q = \Omega \times (0,T)$ and $\sum = \Gamma \times (0,T)$. We are interested in the following boundary value problem for the Oseen equations:

$$\frac{\partial u}{\partial t} - v\Delta u + (z.\nabla)u + (u.\nabla)z + \nabla p = 0 ,$$

$$div\, u = h \quad in\ Q \qquad \text{.......... (1.1)}$$

u = g on $\sum$, **u** (0) = $\boldsymbol{u}_0$ in Ω,

where h and g are non homogeneous terms in the divergence and boundary conditions , and $\boldsymbol{u}_0$ is the initial condition . the viscosity coefficient v is positive $L^\infty(0, \infty; (H^S(\Omega))^N)$ with $s > 1/2$.we are also interested in similar problems for the Navier-Stokes equations .From the divergence theorem it follows that h and g must satisfy the compatibility condition

$$\int_\Omega h(.,t) = \int_\Gamma g(.,t).n(.) \quad \text{for } t \in (0,T)$$

A Classical way for studying equation(1.1) is to consider the solution (w(t), $\pi(t)$) to equation

$$\lambda_0 w(t) - v\Delta w(t) + (z.\nabla)w(t) + (w(t).\nabla)z$$

$$+\nabla\, \pi(t) = 0, div\ w = h(t)\ in\ \Omega$$

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W(t) = g(t) on Γ,

For some $\lambda_0 > 0$ large enough ,and next to look for (u,p) in the form

(u,p) = (w, π) + (y, p).

This method ,that we refer as the lifiting method in this paper,is helpful if h and g are regular enough ,For example if $h \in H^1$ (0,T;$L^2(\Omega)$) and $g \in H^1(0,T;\ H^1;\ \mathbb{R}^N)$). Motivated control problems and related questions ,we would like to study equation (1.1) when $h \in H^1$ (0,T;$L^2(\Omega)$) and $g \in H^1(0,T;\ H^1;\ \mathbb{R}^N)$)concerns the existence of regular solutions for the stokes equation.

2.STOKES AND NAVIER STOKES EQUATION WITH NONHOMOGENEOUS DIVERGENCE CONDITION

In this chapter, we discuss about the some theorems on stokes and navier –stokes equation with non homogeneous divergence condition.

Theorem 2.1

For all (g,h) ∈ $H^1(0,T;H^{-1/2}_{\Gamma}, {}^{-1}_{\Omega})$, and u_0 $\epsilon\ L^2$ (Ω), equation admits a unique weak solution in the sense of (u,p) = (y.q) +(w,π). it is satisfies

$$\| u \|_{L^2\left(0,T;L^2(\Omega)\right)} + \| p \|_{L^2(0,T;\mathcal{H}^{-1}(\Omega))}$$

$$\leq C\left(\| p\boldsymbol{u}_0 \|_{v_n^0(\Omega)} + \| (g,h) \|_{(0,T;H^{1/2}_{\Gamma}, {}^{-1}_{\Omega})}\right)$$

Proof

Given:

For all (g,h) ∈ $H^1(0,T;H^{-1/2}_{\Gamma}, {}^{-1}_{\Omega})$ and u_0 $\epsilon\ L^2$ (Ω), equation admits a unique weak solution

solution in the sense of (u,p) = (y.q) +(w,π)

To prove:

The given equation satisfies,

$$\| u \|_{L^2(0,T;L^2(\Omega))} + \| p \|_{L^2(0,T;\mathcal{H}^{-1}(\Omega))}$$
$$\leq C\left(\| p\boldsymbol{u}_0 \|_{v_n^0(\Omega)} (g,h) \|_{H^1(0,T;H_{\Gamma}^{1/2}, _{\Omega}^{-1})}\right)$$

Let (g,h) be in $H^1(0,T;H_{\Gamma}^{1/2},_{\Omega}^{-1})$.Due to the solution (w(t),$\pi(t)$ to equation is unique and it satisfies

$$\| w \|_{H^1(0,T;L^2(\Omega))} + \| \pi \|_{H^1(0,T;\mathcal{H}^{-1}(\Omega))}$$
$$\leq C \| (g,h) \|_{H^1(0,T;H_{\Gamma}^{1/2}, _{\Omega}^{-1})}$$

Thus in equation $\frac{\partial w}{\partial t}$ belongs to $L^2(0,T;L^2(\Omega))$ and P $(u_0 w(0))$ belongsto $v_n^0(\Omega)$. the equation admits a unique weak solution which satisfies

$$\| y \|_{W(0,T;v_n^1(\Omega),V^{-1}(\Omega))} + \| q \|_{L^2(0,T;\mathcal{H}^0(\Omega))}$$
$$\leq C(\| P(\boldsymbol{u}_0 - w(0)) \|_{v_n^0(\Omega)} + \| w' \|_{L^2(0,T;L^2(\Omega))}).$$

The estimate for (u,p) can be deduced from the estimate obtained for (y.q)and (w,π).

Hence the proof.

Theorem 2.2

For all (g,h) $\in L^2(0,T;H_{\Gamma}^{1/2},_{\Omega}^{-1})$,and $u_0 \in H^{-1}(\Omega)$, equation

$$\frac{\partial u}{\partial t} - v\Delta u + \nabla p = 0. div\, \boldsymbol{u} = h \quad in\ Q,$$

admits a unique solution in the sense of .if satisfies

$$\| u \|_{L^2(Q)}$$
$$\leq C\left(\| u_0 \|_{H^{-1}}(\Omega) + \| (g,h) \|_{L^2(0,T;H_{\Gamma}^{1/2},_{\Omega}^{-1})}\right)$$

Moreover, there exists a distribution $p \in D'(Q)$ such that

$$\frac{\partial u}{\partial t} - v\Delta u + \nabla p = 0 \quad in\ D'(Q),$$

and **u** obeys

div **u** =h in $L^2(0.T;H^{-1}(\Omega))$.

proof:

Given:

For all (g,h) $\in L^2(0,T;H_{\Gamma}^{1/2},_{\Omega}^{-1})$, and $u_0 \in H^{-1}(\Omega)$, equation $\frac{\partial u}{\partial t} - v\Delta u + \nabla p = 0. div\, \boldsymbol{u} = h \quad in\ Q$ admits a unique solution .

To prove:

If satisfies

$$\| u \|_{L^2(Q)} \leq C\begin{pmatrix} \| u_0 \|_{H^{-1}_{(\Omega)}} \\ + \| (g,h) \|_{L^2(0,T;H_{\Gamma}^{1/2},_{\Omega}^{-1})} \end{pmatrix}$$

More over ,there exists a distribution $p \in D'(Q)$ such that

$$\frac{\partial u}{\partial t} - v\Delta u + \nabla p = 0 \quad in\ D'(Q),$$

and **u** obeys

div **u** =h in $L^2(0.T;H^{-1}(\Omega))$.

To prove the uniqueness, we assume that $u_0 = 0$, g = 0 and h=0.In that case ,if **u** is a solution to

$$\frac{\partial u}{\partial t} - v\Delta u + \nabla p = 0. div\, \boldsymbol{u} = h \quad in\ Q$$

In the sense of transposition , by setting **f = u** ,we deduce that **u** = 0. This the uniqueness isestablished.Let us denote by A mapping

$$A = f \mapsto \left(\Phi(0), -v\frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{n}} + \psi\boldsymbol{n} + k(\boldsymbol{\Phi},\psi)\boldsymbol{n}, -\psi - k(\boldsymbol{\Phi},\psi)\right),$$

Where $(\boldsymbol{\Phi},\psi)$ is the solution to equation ,we can easily see that A is bounded operator from $L^2(Q)$ into $H_0^1(\Omega) \times L^2(0,T;H_{\Gamma}^{1/2},_{\Omega}^{1})$.

Thus A^* is abounded operator from

$H^{-1}(\Omega) \times L^2(0,T;H_{\Gamma}^{-1/2},_{\Omega}^{-1})$. into $L^2(Q)$.If we see **u** = $A^*(g,h)$ we have proved the existence of

function**u** $\in L^2(Q)$. Thus we have proved the existence of a function u $\epsilon L^2(Q)$ which satisfies transposition.

Let ψ be in $L^2(0,T;H^{-1}(\Omega))$. In the case $(\boldsymbol{\Phi},\psi)$,with $\boldsymbol{\Phi} = 0$, is the solution to equation

$$\frac{-\partial \boldsymbol{\Phi}}{\partial t} - v\Delta\boldsymbol{\Phi} + \nabla\psi = f, \text{ div } \boldsymbol{\Phi} = 0 \text{ in Q},$$
$$\boldsymbol{\Phi} = 0 \text{ on } \sum, \quad \boldsymbol{\Phi}(\mathrm{T}) = 0 \quad \text{in } \Omega$$

Corresponding to f = $\nabla\psi$. By choosing f = $\nabla\psi$ in transposition, thus we have prove that

$$\int_Q \boldsymbol{u}.\nabla\psi + \langle\psi,h\rangle_{L^2(0,T;H^1(\Omega))L^2(0,T;H^{-1}(\Omega))')}$$
$$=\langle\gamma_0\psi\mathbf{n},\mathbf{g}\rangle_{\langle\psi,h\rangle}{}_{L^2(0,T;H^{1/2}(\Gamma))L^2(0,T;H^{1/2}(\Gamma))},$$

For all $\psi \in L^2(0,T;H^{-1}(\Omega))$ Choosing ψ in $L^2(0,T;H_0^1(\Omega))$ .we obtain div u = h in $L^2(0,T;H^{-1}(\Omega))$. Let $\boldsymbol{\Phi}$ be in $(\mathrm{D(Q)})^{\mathrm{N}}$ such that div $\boldsymbol{\Phi} = 0$. The pair $(\boldsymbol{\Phi},\psi)$, with $\psi = 0$, is the solution to equation .

$$\frac{-\partial \boldsymbol{\Phi}}{\partial t} - \nu\Delta\,\boldsymbol{\Phi} + \nabla\,\psi = f,\ \ \text{div}\ \boldsymbol{\Phi} = 0 \quad \text{in Q,}$$

$$\boldsymbol{\Phi} = 0 \text{ on } \sum,\ \boldsymbol{\Phi}(\mathrm{T}) = 0 \quad \text{in } \Omega$$

Corresponding to $\mathbf{f} = -\partial_t\boldsymbol{\Phi} - \nu\Delta\boldsymbol{\Phi}$. with this choice for in transposition ,we prove that

$$\langle \partial_t \boldsymbol{u} - \nu\Delta\boldsymbol{u}\,, \boldsymbol{\Phi}\,\rangle_{(D'(Q))^N,\,(D\,(Q))^N}$$

$$= \int_Q \mathbf{u}\,(-\partial_t\,\boldsymbol{\Phi} - \ \nu\Delta\,\boldsymbol{\Phi}) = 0,$$

For all $\boldsymbol{\Phi} \in (\mathrm{D(Q)})^{\mathrm{N}}$ such that p such that

$\frac{\partial u}{\partial t} - \nu\Delta u + \nabla p = 0\ \ in\ \ D'(\mathrm{Q})$ is satisfied

Hence the proof.

Theorem 2.3

For all ($\mathbf{g}$,h) $\in H^1\left(0,\mathrm{T}; H\ {}^{1/2}_{\Gamma}\ {}^{-1}_{\Omega}\right)$, and $u_0 \in L^2(\Omega)$,**u** is a solution to equation $\frac{\partial u}{\partial t} - \nu\Delta u + \nabla p = 0.\,div\,\boldsymbol{u} = h\ \ in\ Q$ in the sense of function if and only if it is a solution in the sense of transposition **.**

Proof

Given

For all ($\mathbf{g}$,h) $\in H^1\left(0,\mathrm{T}; H\ {}^{1/2}_{\Gamma\,,}\ {}^{-1}_{C}\right)$, and $u_0 \in L^2(\Omega)$,**u** is a solution to equation $\frac{\partial u}{\partial t} - \nu\Delta u + \nabla p = 0.\,div\,\boldsymbol{u} = h\ in\ Q$ in the sense of function.

To prove

We have If and only if it is a solution in the sense of transpositionWe first establish the theorem in the case when

$$(g,h) \in C^1([0,T]; H^{3/2}_{\Gamma}\,{}^{1}_{\Omega})$$

and p(u_0 $-L(g(0),h(0)))\ \epsilon\ V_0^1(\Omega)$. in that case , the soluation ($\mathbf{u}$,p) =(w,π) $+ (y,q) to$ equation

$$\frac{\partial u}{\partial t} - \nu\Delta u + \nabla p = 0.\,div\,\boldsymbol{u} = h\ \ in\ Q$$

In the sense of weak solution is such that w $\in C^1([0,T];\ H^2(\Omega))$,$\pi\ \epsilon\ C^1([0,T];\ \mathcal{H}^2(\Omega))$.

$\boldsymbol{y} \in \boldsymbol{V}^{2,1}$(Q), And q $\in L^2(0.T;\ \mathcal{H}^2(\Omega))$. Therefore, we can easily verfy that **u** is a solution to equation

$$\frac{\partial u}{\partial t} - \nu\Delta u + \nabla p = 0.\,div\,\boldsymbol{u} = h\ \ in\ Q$$

In the sense of transposition .since the solution to equation in the sense of transposition is unique, the theorem is proved in that case.

Now assume that (($\mathbf{g}$, h) $\in H^1\left(0,\mathrm{T}; H\ {}^{-1/2}_{\Gamma\,,}\ {}^{0}_{\Omega}\right)$, and $\boldsymbol{u}_0 \in (\Omega)$ Consider a sequence

$\{(g_k\,,h_k)\} \subset C^1([0,T]; H^{3/2}_{\Gamma}\,{}^{1}_{\Omega})$ covering to

($\mathbf{g}$, h) $\in H^1\left(0,\mathrm{T}; H\ {}^{-1/2,}_{\Gamma\,,}\ {}^{0}_{\Omega}\right)$,and a sequence

$\{\ \boldsymbol{u}_{0.k}\,\} \subset H^1(\Omega)$ covering to $in\ L^2(\Omega)$.and such that $\{p_{u_{0,k}} - L(g_k\ (0), h_k(0))\} \subset V_0^1(\Omega)$

Converges to (p_{u0} $-$L (g (0), h (0))) in $v_n^0(\Omega)$. let u_k be the solution to the equation

$$\frac{\partial u}{\partial t} - \nu\Delta u + \nabla p = 0.\,div\,\boldsymbol{u} = h \qquad in\ Q$$

Corresponding to (g_k, h_k) and $u_{0,k}$.According to step 1, the solution in the sense of weak solution and transposition coincide. From

$$\|\ u\ \|_{L^2\left(0,T;L^2(\Omega)\right)} + \|\ p\ \|_{L^2(0,T;\mathcal{H}^{-1}(\Omega))}$$

$$\leq\ C\left(\|\ p\boldsymbol{u}_0\ \|_{v_n^0(\Omega)} + \|\ (g,h)\ \|_{H^1(0,\mathrm{T};H\ {}^{-1/2}_{\Gamma\,,}\ {}^{-1}_{\Omega})}\right)$$

It follows that $\{u_k\}$. Converges in $\boldsymbol{L}^2$(Q) to the solution **u** to the equation

$$\frac{\partial u}{\partial t} - \nu\Delta u + \nabla p = 0.\,div\,\boldsymbol{u} = h \qquad in\ Q$$

To the sense of weak solution and from

$$\|\ \ u\ \|_{L^2(Q)} \leq C\left(\|\ u_0\ \|_{H^{-1}_{(\Omega)}} + \|\ (g,h)\ \|_{L^2\left(0,\mathrm{T};H\ {}^{1/2}_{\Gamma\,,}\ {}^{-1}_{\Omega}\right)}\right)$$

It follows that $\{u_k\}$ converges in L^2(Q) to the solution of equation

$$\frac{\partial u}{\partial t} - \nu\Delta u + \nabla p = 0.\,div\,\boldsymbol{u} = h \qquad in\ Q$$

In the sense of transposition .

Hence the proof.

3.STABLISED FINITE ELEMENT METHODS FOR THE

STOKES PROBLEM

In this chapter, we discuss about the some theorems on stabilised finite element methods for the stokes problem.

Theorem 3.1

Assume that either $S_h \subset V_h$ or $p_h \subset C^{\circ}(\Omega)$ and that $0 < \alpha <$ C$_{I.}$ For the approximate solution obtained with Method I we then have

$\|\ u - u_h\ \|_1 + \|\ p -\ p_h\ \|_0 \leq$ C $(H^k\ |\boldsymbol{u}|_{k+1} + h^{l+1}\ |\ p|_{l+1}\ )$,

Provided the exact solution satisfies $\boldsymbol{u} \in [H^{k+1}(\Omega)]^N$ and $p\ \in\ H^{l+1}(\Omega)$.

Proof

Given

Assume that either $S_h \subset V_h$ or $p_h \subset C^{\circ}(\Omega)$ and that 0< $\alpha < C_I$. For the approximate solution obtained with Method I.

To prove

we have

$\| u - u_h \|_1 + \| p - p_h \|_0 \le C\,(H^k\,|\boldsymbol{u}|_{k+1} + h^{l+1}\,|\,p|_{l+1}\,)$,

Provided the exact solution satisfies $\boldsymbol{u} \in [H^{k+1}(\Omega)]^N$ and $p \in H^{l+1}(\Omega)$. Le t $\tilde{\boldsymbol{u}} \in \boldsymbol{V}_h$ be the interpoland of **u** and let $\tilde{p} \in P_h$ be the interpoland of p. We note that now implies the existence of (v,q) ∈ $V_h \times p_h$. such that

$$\| \boldsymbol{v} \|_1 + \| q \|_0 \le C \qquad (3.1.1)$$

and

$$\| \tilde{\boldsymbol{u}} - \boldsymbol{u_h} \|_1 + \| \tilde{p} - p_h \|_0 \le B_h\,(\boldsymbol{u}_h\, v - \tilde{\boldsymbol{u}}, \tilde{p}; \mathbf{v}, q) \qquad (3.1.2)$$

The consistency

$B_h\,(\boldsymbol{u}, p; \boldsymbol{v}, q) = F_h\,(\boldsymbol{v}, q) \quad \forall\ (\boldsymbol{v}, q) \in \boldsymbol{V}_h \times p_h.$

Now gives

$(\boldsymbol{u}_h - \tilde{\boldsymbol{u}}, p_h - \tilde{p}; \mathbf{v}, q) = B_h\,(\boldsymbol{u} - \tilde{\boldsymbol{u}}, p_h - \tilde{p}; \mathbf{v}, q)$

For the right hand side above Schwarz in equality gives

$$(\boldsymbol{u} - \tilde{\boldsymbol{u}}, p - \tilde{p}; \mathbf{v}, q) \le C(\boldsymbol{u} - \tilde{\boldsymbol{u}} \|_1^2 + \sum_{k\in c_h} h_k^2 \| \Delta(\boldsymbol{u} - \tilde{\boldsymbol{u}}) \|_{0,k}^2 + \| p - \tilde{p} \|_0^2 + \sum_{k\in c_h} h_k^2 \| = (p - \tilde{p}) \|_{0,k}^2)^{1/2} \ .(\|\mathbf{v}\|_1^2 + \sum_{k\in c_h} h_k^2 \ \| \Delta v \|_{0,k}^2 \ \| q \|_0^2 + \sum_{k\in c_h} h_k^2 \| \nabla q \|_{0,k}^2)^{1/2}$$

Hence, the in equality ,the inverse in equality

$\sum_{k\in c_h} h_k^2 \| \Delta v \|_{0,k}^2 \le \| \Delta v \|_0^2 \quad \forall\, \mathbf{v} \in \boldsymbol{V}_h.$

$$S \le C \| q \|_0,$$

Together with (1) to (4) give

$$\| \boldsymbol{u} - \boldsymbol{u}_h \|_1 + \|p - ph \|_0 \le C\ (\| \boldsymbol{u} - \tilde{\boldsymbol{u}} \|_1 + \sum_{k\in c_h} h_k^2 \| \Delta(\boldsymbol{u} - \tilde{\boldsymbol{u}}) \|_{0,k}^2 + \| p - \tilde{p} \|_0 + (\sum_{k\in c_h} h_k^2 \| \nabla(p - \tilde{p}) \|_{0,k}^2)^{1/2}$$

The following interpolation estimate is standard

$$\| \boldsymbol{u} - \tilde{\boldsymbol{u}} \|_1 + (\sum_{k\in c_h} h_k^2 \| \Delta(\boldsymbol{u} - \tilde{\boldsymbol{u}}) \|_{0,k}^2)^{1/2} \le C\, h^k\, |\, \boldsymbol{u}\, |_{k+1} + h^{l+1}\, |p|_{l+1}\,).$$

Hence the proof

Theorem 3.2

Suppose that either one of the assumptions

$$\boldsymbol{S}_h \subset \boldsymbol{V}_h \text{ or } p_h \subset C^{\circ}(\Omega)$$

Is valid then there exist positive constants C_1 and C_2 such that

$$\sup_{0 \ne v \ne \in V_h} \frac{(div\ \boldsymbol{v}, p)}{\|V\|_1} \ge C_1\ \| p \|_0 - C_2\ ((\sum_{k\in c_h} h_k^2 \|_{0,k}^2)^{1/2} \quad \forall_P \in P_h.$$

Proof

Given

Suppose that either one of the assumptions

$$\boldsymbol{S}_h \subset \boldsymbol{V}_h \text{ or } p_h \subset C^{\circ}(\Omega)$$

is veiled Then there exist positive constants C_1 and C_2 .

To prove

Then there exist positive constants C_1 and C_2 such that

$$\sup_{0 \ne v \ne \in V_h} \frac{(div\, \boldsymbol{v}, p)}{\| V \|_1} \ge C_1\ \| p \|_0 - C_2\ ((\sum_{k\in c_h \|} h_k^2 \|)^{1/2} \quad \forall_P \in P_h.$$

Consider first the case $\boldsymbol{S}_h \subset \boldsymbol{V}_h$. denote by Π_h the L^2 projection onto the space of piecewise constants

$$\Pi_h q|k = \frac{1}{area\,(k)} \int_k q d\Omega, \qquad \forall\, k \in C_h\ .$$

Since $\boldsymbol{S}_h \subset \boldsymbol{V}_h$ the pair (V_h, $\Pi_h\, p_h$) satisfies the stability in equality and Raviart . Hence there is a constant C_1 such that for every $p \in P_h$ there exsist **v** $\in \boldsymbol{V}_h$, with $\| V \|_1 = 1$, such that

$$(\text{div}\ \mathbf{v}, \Pi_h\ p\ ) \ge C_1\ \| \Pi_h p \|_0\, .$$

Using the interpolation estimate

$$\| (I - \Pi_h) p \|_{0,k}\, . \quad \forall\, k \in C_h\ ,$$

We now that

$$(\text{div}\ \mathbf{v}, p\ ) = (\text{div}\ \mathbf{v}, \Pi_h p\ (\text{div}\ \mathbf{v}, \| (I - \Pi_h) p)\ \|$$

$$\ge C_1\ \| \Pi_h p \|_0 - \| V \|_1 \| (I - \Pi_h) p \|_0$$

$$\ge C_1\ \| \Pi_h p \|_0 - \| (I - \Pi_h) p \|_0$$

$$\ge C_1\ \| p \|_0 - (1 + C_1)\ \|(I - \Pi_h)\, p \|_0$$

$$\ge C_1\ \| p \|_0 - (1 + C_1)\ C_3\ (\sum_{k\in c_h} h_k^2 \|_{0,k}^2)^{1/2}$$

And the asserted estimate with the first assumption is proved.Next let us consider the case $p_h \subset C^{\circ}(\Omega)$.

Since p $\in P_h \subset L_O^2(\Omega)$,By the continuous version of the ''inf-sup'' condition, there is a non-vanishing $\mathbf{W} \in [H_0^1(\Omega)]^N$ such that

$$\text{div}(\boldsymbol{w}, p) \geq C_4 \| p \|_0 \| \boldsymbol{w} \|_1 .$$

further ,one can show that there is an interpolate $\widetilde{\boldsymbol{w}} \in V_h$ to $\mathbf{w}$ such that

$$\sum_{k \in c_h} h_k^{-2} \| \boldsymbol{w} - \widetilde{\boldsymbol{w}} \|_{0,k}^2 \leq C_5 \| \boldsymbol{w} \|_1$$

And

$$\| \widetilde{w} \|_1 \leq C_6 \| w \|_1 .$$

Integrating by parts on each $k \in c_h$, and using the above estimates we get

$$\text{div}(\mathbf{w}, p) = (\text{div}(\widetilde{\boldsymbol{w}} - \boldsymbol{w}), p) + (\text{div}\, \mathbf{w}, p)$$

$$\geq (\text{div}(\widetilde{\boldsymbol{w}} - \boldsymbol{w}), p) + C_4 \| \boldsymbol{w} \|_1 \| p \|_0$$

$$\geq \{-C_5 (\textstyle\sum_{k \in c_h} h^2 k \| \nabla p \|_{0,k}^2) C_4 \| p \|_0 \} \| \boldsymbol{w} \|_1 .$$

Dividing by $\| \boldsymbol{w} \|_1$ we obtain

$$\frac{\text{div}(\widetilde{\boldsymbol{w}}, p)}{\| \boldsymbol{w} \|_1} \geq C_4 \| p \|_0 - C_5 \textstyle\sum_{k \in c_h} h^2 \text{k}$$

$$\| \nabla p \|_{0,k}^2)^{1/2}$$

And (1) gives

$$\frac{\text{div}(\widetilde{\boldsymbol{w}}, p)}{\| \boldsymbol{w} \|_1} \geq \frac{\text{div}(\widetilde{\boldsymbol{w}}, p)}{C_6 \| \boldsymbol{w} \|_1} \geq C_6^{-1} \left\{ C_4 \| p \|_0 - C_5 . \textstyle\sum_{k \in c_h} h^2 \text{k} \| \nabla p \|_{0,k}^2)^{1/2} \right\}$$

Hence the proved.

Theorem 3.3

Assume that either $S_h \subset V_h$ or $p_h \subset C^\circ(\Omega)$ and that $\alpha > 0$.For Method H we then have the following error estimate

$$\| \text{u} - \boldsymbol{u}_h \|_1 + p - p_h \|_0 \leq C(h^k |\mathbf{u}|_{k+1} + h^{l+1} | p |_{l+1}).$$

Proof:

Given:

Assume that either $S_h \subset V_h$ or $p_h \subset C^\circ$ (Ω) and that $\alpha > 0$.For Method H

To prove:

We then have the following error estimate

$$\| \text{u} - \boldsymbol{u}_h \|_1 + p - p_h \|_0 \leq C(h^k |\mathbf{u}|_{k+1} + h^{l+1} | p |_{l+1}).$$

The analysis differs from that of earlier method only with respect to the verification of the stability.

To this end let $\gamma > 1$ be parameter and estimate

$$B_h(\mathbf{u}, p; \mathbf{c}, -p) = \| \nabla \mathbf{u} \|_0^2 + \alpha h_k^2 \| -\Delta \mathbf{u} + \nabla \text{p} \|_{o,k}^2$$

$$= \| \nabla \mathbf{u} \|_0^2 + \alpha c \nabla \mathbf{u} \|_{o,k}^2$$

$$+2 \alpha \textstyle\sum_{k \in c_h} h_k^2 (-\Delta \mathbf{u}, \nabla \text{p})_k$$

$$+ \alpha \textstyle\sum_{k \in c_h} h_k^2 \| \nabla \text{p} \|_{o,k}^2$$

$$\geq \| \nabla \mathbf{u} \|_0^2 + \alpha(1-\gamma) \textstyle\sum_{k \in c_h} h_k^2 \| \Delta \mathbf{u} \|_{o,k}^2$$

$$+(1 - \frac{1}{\gamma}) \alpha \textstyle\sum_{k \in c_h} h_k^2 \| \nabla \text{p} \|_{o,k}^2$$

$$\geq (1 + \alpha(1-\gamma) C_I^{-1}) \| \nabla \mathbf{u} \|_0^2$$

$$+(1 - \frac{1}{\gamma}) \alpha \textstyle\sum_{k \in c_h} h_k^2 \| \nabla \text{p} \|_{o,k}^2$$

$$\geq C_2 \| \nabla \mathbf{u} \|_0^2 + C_3 \alpha \textstyle\sum_{k \in c_h} h_k^2 \| \nabla \text{p} \|_{o,k}^2$$

$$\geq C(\| \nabla \mathbf{u} \|_0^2 + \textstyle\sum_{k \in c_h} h_k^2 \| \nabla \text{p} \|_{o,k}^2)$$

When we chose $1 - \gamma\ (1 + C_I^{-1})$. With the stability is now proved.

Theorem 3.4

Assume that either $S_h \subset V_h$ or $p_h \subset C^\circ(\Omega)$ and that $\alpha > 0$. for the approximation with method (Franca and Stenberg [1991]. we then have the following error estimate

$$\| \sigma - \sigma_h \|_0 + \| p - p_h \|_0 + \| \text{u} - u_h \|_1$$

$$\leq C(h^{m+1} |\sigma|_{m+1} + h^k |\mathbf{u}|_{k+1},$$

For all $\in$ in the range

$$0 \leq \in \leq Ch^s, s = \min \{k+1, l+2\}.$$

Proof

Given

Assume that either $S_h \subset V_h$ or $p_h \subset C^\circ(\Omega)$ and that $\alpha > 0$. for the approximation with method (Franca and Stenberg [1991].

To prove:

We then have the following error estimate

$$\| \sigma - \sigma_h \|_0 + \| p - p_h \|_0 + \| \text{u} - u_h \|_1$$

$$\leq C(h^{m+1} |\sigma|_{m+1} + h^k |\mathbf{u}|_{k+1},$$

For all $\in$ in the range

$$0 \leq \in \leq Ch^s, s = \min \{k+1, l+2\}.$$

We will not given the proof . we first note that the method is consistent. Hence, In analogy with the earlier methods we have to verify the stability condition which now is

$$\sup_{\substack{(\tau,q,v) \in \sum h \times p_h \times v_h \\ (\tau,q,v) \neq (0,0,0)}} \frac{B_h(\sigma,p,u,\tau,q.v)}{\|\tau\|_0 + \|q\|_0 + \|v\|_1} \geq C(\| \sigma \|_0$$

$$\| p \|_0 + \| \text{u} \|_1) \ \forall\ (\sigma, p, \boldsymbol{u}) \in \textstyle\sum_h \times\ p_h\ V_h V_h$$

To prove this we first use an inverse estimate similar

$C_I \sum_{K\in C_h} h_k^2 \parallel \Delta\mathbf{v}\parallel_{o,k}^2 \leq \parallel \Delta\mathbf{v}\parallel_0^2 \qquad \forall\, v \in V_h.$

And estimate as in order to get

$B_h\ (\sigma, p, \boldsymbol{u}\ ;-\sigma, -p, \boldsymbol{u}) = \parallel \sigma \parallel_0^2 + \epsilon \parallel p \parallel_0^2$

$+\alpha \sum_{k\in c_h} h_k^2 \parallel \mathbf{div} - \nabla p \parallel_{o,k}^2$

$\geq C_1 (\parallel \sigma \parallel_o^2 + \sum_{k\in c_h} h_k^2 \parallel \nabla p \parallel_{o,k}^2)$

$+ \epsilon \parallel p \parallel_0^2.$ (3.4.1)

The second step is to use in the same way as earlier and conclude that there is a (velocity $W \in V_{h,}$ with $\parallel w \parallel_1 \leq \parallel p \parallel_0$, such that

$B_h\ (\sigma, p, \boldsymbol{u}\ ;\mathbf{0},0, ;-w) \geq C_2 \parallel p \parallel_0^2 - C_3 (\parallel \sigma \parallel_o^2$

$+ \parallel \nabla u \parallel_0^2 + \sum_{k\in c_h} h_k^2 \parallel \nabla p \parallel_{o,k}^2)$ (3.4.2)

Next ,the assumption that m = k – 1 for triangles that there is

$\kappa \in \sum_h$ such that $(\kappa, \nabla u) = \parallel \nabla u \parallel_0^2$ (3.4.2a)

And $\parallel \kappa \parallel \leq C \parallel \nabla u \parallel_0$. (3.4.2b)

This gives

$B_h\ (\sigma, p, \boldsymbol{u}\ ;\mathbf{0},0) \geq \parallel \nabla u \parallel_0^2 - C_5 (\parallel \sigma \parallel_o^2$

$+ \sum_{k\in c_h} h_k^2 \parallel \nabla p \parallel_{o,k}^2)$ (3.4.3)

The stability estimate is now obtained from (1) ,(2) and (3) by taking

$(\tau, q, v) = -\sigma + \delta k, -p, u - \delta^2 w)$

With δ positive and small enough.

CONCLUSION

Throughout this work we discussed some definition and theorems on stokes and navier-stokes equation with a nonhomogeneous divergence condition .and then we discussed a stabilised finite element methods for the stokes problem. finally we establish that, weak solution to equation and regularity of solution to the stokes and oseen equation with a non homogeneous divergence condition.

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