VAGUE SMOOTH FUZZY TOPOLOGICAL SPACES

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Abstract: The purpose of this paper is to introduce the concept of vague smooth fuzzy topological spaces by using vague smooth fuzzy sets and to study some of their fundamental properties. Also vague fuzzy continuity, vague fuzzy compactness, vague fuzzy almost compactness, vague fuzzy nearly compactness, vague fuzzy S-closed spaces, vague fuzzy Ti (i = 1, 2, ½) spaces and vague fuzzy normal spaces are studied.

Keywords: Vague fuzzy continuity, vague fuzzy strong continuity, vague fuzzy almost continuity, vague fuzzy weakly continuity, vague fuzzy compactness, vague fuzzy nearly compactness, vague fuzzy S-closed space, vague fuzzy Ti (i = 1, 2, ½) spaces and vague fuzzy normal spaces.

1. Introduction and Preliminaries:

Vague sets provide an intelligent search to find the most suitable match to answer any imprecise query made by the database users. Consequently, there is a genuine necessity of a model like vague sets, a kind of higher order fuzzy set. The concept of vague sets was first introduced by Gau and Buehrer[2]. This provides a natural framework to develop vague smooth fuzzy topological spaces by generalizing the concepts of intuitionistic fuzzy topology.

The purpose of this paper is to introduce the concept of vague smooth fuzzy topological spaces in the sense of Šostak [3]. Some properties concerning vague fuzzy continuity and vague fuzzy compactness are studied using the notions of intuitionistic fuzzy compactness introduced by Abbas [1] and compactness in intuitionistic fuzzy topological spaces introduced by Ramadan, Abbas and Abd El-Latif [4]. Further, vague fuzzy Ti (i = 1, 2, ½) spaces are introduced and its properties are studied using the concept introduced by Ramadan, Abbas and Abd El-Latif [5]. In addition vague fuzzy normal spaces are introduced using the notion of Tomasz Kubiaik [6].

Definition 1.1. [2]

Let X be a non-empty set. A vague set A in X is characterized by two membership functions given by:

i) a truth membership function \( t_A : X \rightarrow [0,1] \) and

ii) a false membership function \( f_A : X \rightarrow [0,1] \)

The grade of membership of x in the vague set A is bounded by a sub-interval \([t_A(x), 1 - f_A(x)]\) of \([0,1]\). This indicates that if the actual grade of membership is \( \mu(x) \), then \( t_A(x) \leq \mu(x) \leq 1 - f_A(x) \).

Definition 1.2. [2]

Let X be a non-empty set and Let A, B be vague sets in X. Then

[a] \( A \subseteq B \) iff \( t_A(x) \leq t_B(x) \) and \( f_A(x) \geq f_B(x) \) for every \( x \in X \).

[b] \( A = B \) iff \( t_A(x) = t_B(x) \) and \( f_A(x) = f_B(x) \) for every \( x \in X \).

[c] \( \bar{A} = \{ (x, t_A(x), f_A(x)) / x \in X \} \)

[d] \( A \cap B = \{ (x, \min(t_A(x), t_B(x)), \max(f_A(x), f_B(x))) / x \in X \} \)

[e] \( A \cup B = \{ (x, \max(t_A(x), t_B(x)), \min(f_A(x), f_B(x))) / x \in X \} \)

Definition 1.3. [2]

Let \( \{ A_i : i \in I \} \) be an arbitrary family of vague sets in X. Then

[a] \( \bigcap A_i = \{ x \in X / (x, \min_{i \in I} t_{A_i}(x), \max_{i \in I} f_{A_i}(x)) \} \).

[b] \( \bigcup A_i = \{ x \in X / (x, \max_{i \in I} t_{A_i}(x), \min_{i \in I} f_{A_i}(x)) \} \).

Definition 1.4. [2]

A vague set A of X with \( t_A(x) = 0 \) and \( f_A(x) = 1 \), \( \forall x \in X \) is called a zero vague set of X and is denoted as \( \overline{0} = \langle x, 0, 1 \rangle \).

Definition 1.5. [2]

A vague set A of X with \( t_A(x) = 1 \) and \( f_A(x) = 0 \), \( \forall x \in X \) is called a unit vague set of X and is denoted as \( \overline{1} = \langle x, 1, 0 \rangle \).

Definition 1.6. [2]

A vague set A of X with \( t_A(x) = \alpha \) and \( f_A(x) = 1 - \alpha \), \( \forall x \in X \) is called the \( \alpha \)-vague set of X, where \( \alpha \in [0,1] \).

Throughout this paper, Let X be a non-empty set, \( I = [0,1] \) and \( I_0 = (0,1] \). For \( \alpha \in I \), \( \overline{\alpha}(x) = \alpha \) for all \( x \in X \) and let \( (X, T) \) & \( (Y, S) \) be any two vague smooth fuzzy topological spaces.

2. Vague smooth fuzzy topological spaces

Definition 2.1.

Let X be a non-empty set and \( I \) be the closed interval \([0,1]\). An vague smooth fuzzy set (VFS) A is an object having the form \( A = \{ (x, t_A(x), f_A(x)) : x \in X \} \), where the mappings \( t_A : X \rightarrow I \) and \( f_A : X \rightarrow I_0 \) denote the truth membership function (namely \( t_A(x) \)) and the false membership function (namely \( f_A(x) \)) of each element \( x \in X \) to the set A respectively.

Further \( 0 \leq t_A(x) + f_A(x) \leq 1 \). The complement of the VFS A, is \( \overline{A} = \{ (x, f_A(x), t_A(x)) : x \in X \} \).

Notation 2.1. For a given non-empty set X, denote the vague smooth fuzzy topology on X by \( \mathcal{T} \).
family of VFS's in X by the symbol $\zeta^X$.

**Definition 2.2.**

Let X be a non-empty set and $x \in X$ a fixed element in X. If $r \in I_\rho, s \in I_\lambda$ are fixed real numbers such that $r + s \leq 1$, then the VFS $x_{r,s} = \langle y, x_n, 1 - x_{n+1} \rangle$ is called a vague fuzzy point (VFP) in X, where r denotes the truth membership function of $x_{r,s}$, $s$ the false membership function of $x_{r,s}$, and $x \in X$ the support of $x_{r,s}$. The VFP $x_{r,s}$ is contained in the VFS A if and only if $r < t_A(x)$ and $s > f_A(x)$.

**Definition 2.3.**

[a] An VFP $x_{r,s}$ in X is said to be quasicoincident with the VFS A, denoted by $x_{r,s} \in A$, if and only if $s < t_A(x) \land r > f_A(x)$, $x_{r,s} \in A$ if and only if $x_{r,s} \notin A$.
[b] The VFS's A and B is said to be quasicoincident, denoted by $A \equiv B$ if and only if there exists an element $x \in X$ such that $t_A(x) > f_A(x)$ or $f_A(x) < t_A(x)$.

**Definition 2.4.**

Let a and b be two real numbers in $[0,1]$ satisfying the inequality $a + b \leq 1$. Then the pair $(a,b)$ is called an vague fuzzy pair.

Let $(a_1,b_1), (a_2,b_2)$ be any two vague fuzzy pairs. Then define:

[i] $(a_1,b_1) \leq (a_2,b_2)$ if and only if $a_1 < a_2$ and $b_1 \geq b_2$;
[ii] $(a_1,b_1) = (a_2,b_2)$ if and only if $a_1 = a_2$ and $b_1 = b_2$;
[iii] If $\{ a_{i,b_i} ; i \in I \}$ is a family of vague fuzzy pairs, then $\bigvee(a_{i,b_i}) = \langle \bigvee a_i, \bigvee b_i \rangle$ and $\bigwedge(a_{i,b_i}) = \langle \bigwedge a_i, \bigwedge b_i \rangle$;
(iv) the complement of an vague fuzzy pair $(a,b)$ is the vague fuzzy pair $(b,a)$;
[v] $\tilde{1} = \langle 1,0 \rangle$ and $\tilde{0} = \langle 0,1 \rangle$.

**Definition 2.5.**

A function $T : \zeta^X \rightarrow I \times I$ is called a vague smooth fuzzy topology on X if it satisfies the following conditions:

[a] $T(\tilde{0}) = \tilde{T}(\tilde{1}) = \tilde{1}$;
[b] $T(A \land B) \geq T(A) \land T(B)$ for any $A, B \in \zeta^X$;
[c] $T(\bigcup B_i) \geq \bigvee \{ T(B_i) \} \land T(A)$ for any $B_i, A \in \zeta^X$.

The pair $(X,T)$ is called a vague smooth fuzzy topological space. For any $A \in \zeta^X$, the number $t_f(A)$ is called the openness degree of A, while $f_f(A)$ is called the nonopeneness degree of A.

**Definition 2.6.**

Let $(X,T)$ be a vague smooth fuzzy topological space. Then the VFF $T^*$ is defined by $T^*(A) = \tilde{T}(\tilde{1} - A)$. The number $t_{f^*}(A) = t_f(\tilde{1} - A)$ is called the closedness degree of A, while $f_{f^*}(A) = f_f(\tilde{1} - A)$ is called the nonclosedness degree of A.

**Proposition 2.1.** The VFF $T^*$ on X satisfies the following conditions:

[a] $T^*(\tilde{0}) = T^*(\tilde{1}) = \tilde{1}$;
[b] $T^*(A \lor B) \geq T^*(A) \lor T^*(B)$ for any $A, B \in \zeta^X$;
[c] $T(\bigcap B_i) \geq \bigwedge \{ T(B_i) \}$ for any $B_i \in \zeta^X$.

**Definition 2.7.** Let A be an VFS in X. Then the vague fuzzy closure and vague fuzzy interior of A are defined by $C_{\alpha\beta}(A) = \{ B \in \zeta^X : A \subseteq B, T^*(B) \geq (\alpha,\beta) \}$, $I_{\alpha\beta}(A) = \{ B \in \zeta^X : A \subseteq B, T(B) \geq (\alpha,\beta) \}, \alpha \in I_\alpha, \beta \in I_\lambda$.

**Proposition 2.2.** The closure and interior operator satisfy the following properties:

[i] $A \subseteq B$ and $(\alpha,\beta) \preceq (p,q)$ implies $C_{\alpha\beta}(A) \subseteq C_{p,q}(A)$;
[ii] $A \subseteq B$ and $(\alpha,\beta) \preceq (p,q)$ implies $I_{\alpha\beta}(B) \subseteq I_{p,q}(A)$;
[iii] $C_{\alpha\beta}(C_{\alpha\beta}(A)) = C_{\alpha\beta}(A)$;
[iv] $I_{\alpha\beta}(I_{\alpha\beta}(A)) = I_{\alpha\beta}(A)$;
[v] $C_{\alpha\beta}(A \cup B) = C_{\alpha\beta}(A) \cup C_{\alpha\beta}(B)$;
[vi] $I_{\alpha\beta}(A \cap B) = I_{\alpha\beta}(A) \cap I_{\alpha\beta}(B)$;
[vii] $1 - C_{\alpha\beta}(A) = I_{\alpha\beta}(1 - A)$;
[viii] $1 - I_{\alpha\beta}(A) = C_{\alpha\beta}(1 - A)$.

**Definition 2.8.**

Let $f : (X,T) \rightarrow (Y,S)$ be a mapping. Then f is said to be:

[i] vague fuzzy continuous iff $T(f^{-1}(B)) \supseteq S(B)$, for each $B \in \zeta^Y$;
[ii] vague fuzzy open iff $T(A) \subsetneq (\alpha,\beta)$ implies $S(f(A)) \subsetneq (\alpha,\beta)$, for each $A \in \zeta^X$;
[iii] vague fuzzy closed iff $T^*(A) \subseteq (\alpha,\beta)$ implies $S^*(f(A)) \subseteq (\alpha,\beta)$, for each $A \in \zeta^X$.

**Proposition 2.3.** Let $(X,T)$ and $(Y,S)$ be any two vague smooth fuzzy topological spaces and let $f : (X,T) \rightarrow (Y,S)$ be a bijective and vague fuzzy continuous function. Then the following statements are equivalent:

[i] f is a vague fuzzy open function.
[ii] f is a vague fuzzy closed function.
[iii] $(C_{\alpha\beta}(A)) \supseteq C_{\alpha\beta}(f(A))$, for each $A \in \zeta^X$.

3. Vague fuzzy almost continuous and vague fuzzy weakly continuous mapping

**Definition 3.1.** Let A be a VFS in a vague smooth fuzzy topological space $(X,T)$. For $\alpha \in I_\alpha, \beta \in I_\lambda$ with $\alpha + \beta \leq 1$, A is called

[i] $(\alpha,\beta)$- vague fuzzy regular open ($(\alpha,\beta)$-VFRO) if $I_{\alpha\beta}(C_{\alpha\beta}(A)) = A$;
[ii] $(\alpha,\beta)$- vague fuzzy regular closed ($(\alpha,\beta)$-VFRC) if $C_{\alpha\beta}(I_{\alpha\beta}(A)) = A$.
Proposition 3.1. Let A be a VFS in a vague smooth fuzzy topological space (X,T). Then for α ∈ [0, 1], β ∈ [0, 1] with α + β ≤ 1,
[i] if A is (α,β)-VFRO (resp., (α,β)-VFRC) set then T(A) ∋ (α,β) (resp., T*(A) ∋ (α,β)).
[ii] if A is (α,β)-VFSRO set if and only if 1 − A is (α,β)-VFSRO set.

Proposition 3.2. Let (X,T) be a vague smooth fuzzy topological space. Then,
[i] the union of two (α,β)-VFSRO sets is (α,β)-VFSRO set;
[ii] the intersection of two (α,β)-VFSRO sets is (α,β)-VFSRO set.

Definition 3.2. Let f: (X,T) → (Y,S) be a mapping. Then f is said to be
[i] vague fuzzy strong continuous iff T(f⁻¹(A)) = S(f(A)), for each A ∈ C;
[ii] (α,β)-vague fuzzy almost continuous iff T(f⁻¹(A)) ≥ (α,β), for each (α,β)-VFSRO set A ∈ C;
[iii] (α,β)-vague fuzzy weakly continuous iff S(A) ≥ (α,β) implies T(f⁻¹(A)) ≥ (α,β) for each A ∈ C.

Remark 3.1. From the above definitions the following implications are true
(α,β)-Vague fuzzy strong continuous ⇒ vague fuzzy continuous
(α,β)-Vague fuzzy weakly continuous
The converse implications are not true, as shown by the following examples.

Example 3.1. Let X = {a, b, c} and G_J be VFS's in X defined as follows:
G_1 = {α(a, 0.4, 0.1), β(b, 0.6, 0.2), γ(c, 0.5, 0.3)},
G_2 = {α(a, 0.4, 0.4), β(b, 0.4, 0.4), γ(c, 0.4, 0.4)}.

We define VFTs T: C → I x I and S: C → I x I as follows:
\[ T(A) = \begin{cases} 
1 & \text{if } A \supseteq 0.5, 0.2, \\
0.5 & \text{if } A \supseteq 1, 0.2, \\
0 & \text{otherwise,} 
\end{cases} 
\]

Let α = 0.4, β = 0.5. Then the identity mapping id: (X,T) → (X,T) is (α,β)-Vague fuzzy almost continuous but not vague fuzzy continuous.

Example 3.2. Let X = {a, b} and Y = {1, 2} Let G_1 be VFS of X and G_2 be VFS of Y defined as follows: G_1 = {α(a, 0.4, 0.4), β(b, 0.4, 0.4)}, G_2 = {α(1, 0.4, 0.4), β(2, 0.5, 0.4)}.

We define VFTs T: C → I x I and S: C → I x I as follows:
\[ T(A) = \begin{cases} 
1 & \text{if } A \supseteq 0.8, 0.2, \\
0.5 & \text{if } A \supseteq 0.7, 0.1, \\
0 & \text{otherwise,} 
\end{cases} 
\]

Consider the mapping f: (X,T) → (Y,S) defined by f(a) = 1, f(b) = 2. Let α = 0.6, β = 0.3. Then f is (α,β)-Vague fuzzy weakly continuous but not vague fuzzy continuous.

Example 3.3. Let X = {a, b} and Y = {1, 2} Let G_1 be VFS of X and G_2 be VFS of Y defined as follows: G_1 = {α(a, 0.4, 0.4), β(b, 0.4, 0.4)}, G_2 = {α(1, 0.4, 0.4), β(2, 0.5, 0.4)}.

We define VFTs T: C → I x I and S: C → I x I as follows:
\[ T(A) = \begin{cases} 
1 & \text{if } A \supseteq 0.8, 0.2, \\
0 & \text{otherwise.} 
\end{cases} 
\]

Consider the mapping f: (X,T) → (Y,S) defined by f(a) = 1, f(b) = 2. Let α = 0.6, β = 0.3. Then f is (α,β)-Vague fuzzy continuous but not vague fuzzy strong continuous.

4. Various forms of compactness in vague smooth fuzzy topological spaces:

Definition 4.1. Let A ∈ C and α ∈ [0, 1], β ∈ [0, 1] with α + β ≤ 1. Then, A is said to be (α,β)-vague fuzzy compact if and only if for every family \[ \{G_j : T(G_j) \supseteq (α,β), j \in J\} \] with \[ A \subseteq \bigcup_{j \in J} G_j \], there exists a finite subset \[ J_0 \subseteq J \] such that \[ A \subseteq \bigcup_{j \in J_0} G_j \].

Definition 4.2. (X,T) is said to be (α,β)-vague fuzzy compact (resp., (α,β)-vague fuzzy nearly compact) and (α,β)-vague fuzzy almost compact if and only if for every family \[ \{G_j : T(G_j) \supseteq (α,β), j \in J\} \] such that \[ \bigcup_{j \in J} G_j = \overline{A} \], where \[ A \subseteq \bigcup_{j \in J} G_j \] with α + β ≤ 1, there exists a finite subset \[ J_0 \subseteq J \] such that \[ \bigcup_{j \in J_0} G_j = \overline{A} \] (resp., \[ A = \bigcup_{j \in J_0} \overline{G_j} \]).

Proposition 4.1. For α ∈ [0, 1], β ∈ [0, 1] with α + β ≤ 1, (α,β)-vague fuzzy compactness implies (α,β)-vague fuzzy nearly compactness which implies (α,β)-vague fuzzy almost compactness.

Definition 4.3. (X,T) is called (α,β)-vague fuzzy regular if and only if every vague smooth fuzzy set A ∈ C such that T(A) ≥ (α,β), where \[ A \subseteq \bigcup_{j \in J} G_j \] with α + β ≤ 1 can be written as \[ A = \bigcup \{B : T(B) \supseteq T(A), C \subseteq B\} \].

Proposition 4.2. If (X,T) is (α,β)-vague fuzzy almost compact and (α,β)-vague fuzzy regular, then it is (α,β)-vague fuzzy compact.
Proposition 4.3. If (X,T) is (α,β)-vague fuzzy nearly compact and (α,β)-vague fuzzy regular, then it is (α,β)-vague fuzzy compact.

Proposition 4.4. Let \( f : (X,T) \to (Y,S) \) be a vague fuzzy continuous mapping. If \( A \subseteq \mathbb{C}^X \) is (α,β)-vague fuzzy compact, then so is \( f(A) \) in (Y,S), where \( \alpha \in I_0, \beta \in I_1 \) with \( \alpha + \beta \leq 1 \).

Proposition 4.5. Let \( f : (X,T) \to (Y,S) \) be a surjective vague fuzzy continuous mapping. If (X,T) is (α,β)-vague fuzzy compact, then so is \( (Y,S) \), where \( \alpha \in I_0, \beta \in I_1 \) with \( \alpha + \beta \leq 1 \).

Proposition 4.6. Let \( f : (X,T) \to (Y,S) \) be a surjective vague fuzzy continuous mapping. If (X,T) is (α,β)-vague fuzzy almost compact, then so is \( (Y,S) \), where \( \alpha \in I_0, \beta \in I_1 \) with \( \alpha + \beta \leq 1 \).

Proposition 4.7. Let \( f : (X,T) \to (Y,S) \) be a surjective vague fuzzy weakly continuous mapping. If (X,T) is (α,β)-vague fuzzy compact, then so is \( (Y,S) \), where \( \alpha \in I_0, \beta \in I_1 \) with \( \alpha + \beta \leq 1 \).

Lemma 4.1. Let \( V \in \mathbb{C}^X \). Then \( x_{r,s} \in C_{\alpha\beta}(V) \) if and only if for each \( U \in \mathbb{C}^X \), \( \mathbb{T}(U) \geq (\alpha, \beta) \) and \( x_{r,s} \notin U \), where \( r,s \in I_0, \beta \in I_1 \) with \( r + s, \alpha + \beta \leq 1 \).

Lemma 4.2. For \( \alpha \in I_0, \beta \in I_1 \) with \( \alpha + \beta \leq 1 \), \( C_{\alpha\beta}(I_{\alpha\beta}(A)) = C_{\alpha\beta}(A) \), for each \( A \in \mathbb{C}^X \) with \( \mathbb{T}(A) \geq (\alpha, \beta) \).

Definition 4.4.
A family \( \{G_j : T^*(G_j) \geq (\alpha, \beta), j \in J \} \) where \( \alpha \in I_0, \beta \in I_1 \) with \( \alpha + \beta \leq 1 \) has the finite intersection property (F.I.P) if and only if for any finite subset \( J_0 \) of \( J \), \( \bigcap_{j \in J_0} G_j \neq \emptyset \).

Proposition 4.8. \( (X,T) \) is (α,β)-vague fuzzy compact if and only if \( \bigcap_{j \in J} T^*(G_j) \geq (\alpha, \beta), j \in J \) where \( \alpha \in I_0, \beta \in I_1 \) with \( \alpha + \beta \leq 1 \) having the F.I.P. if and only if \( \bigcap_{j \in J} G_j \neq \emptyset \).

Proposition 4.9. \( (X,T) \) is (α,β)-vague fuzzy almost compact if and only if \( \bigcap_{j \in J} I_{\alpha\beta}(G_j) \geq (\alpha, \beta), j \in J \) where \( \alpha \in I_0, \beta \in I_1 \) with \( \alpha + \beta \leq 1 \) having the F.I.P. if and only if \( \bigcap_{j \in J} I_{\alpha\beta}(G_j) \neq \emptyset \).

Proposition 4.10 in \( (X,T) \) the following conditions are equivalent

(i) \( (X,T) \) is (α,β)-vague fuzzy almost compact

(ii) For every family \( \{G_j, j \in J\} \) of (α,β)-VFRO sets such that \( \bigcap_{j \in J} G_j \neq \emptyset \), there exists a finite subset \( J_0 \) of \( J \) such that \( \bigcap_{j \in J_0} I_{\alpha\beta}(G_j) \neq \emptyset \), where \( \alpha \in I_0, \beta \in I_1 \) with \( \alpha + \beta \leq 1 \).

(iii) \( \bigcap_{j \in J} C_{\alpha\beta}(G_j) \neq \emptyset \), holds for every family \( \{G_j, j \in J\} \) of (α,β)-VFRO sets having the F.I.P, where \( \alpha \in I_0, \beta \in I_1 \) with \( \alpha + \beta \leq 1 \).

(iv) For every family \( \{G_j, j \in J\} \) of (α,β)-VFRO sets such that \( \bigcup_{j \in J} G_j = \tilde{1}, \) there exists a finite subset \( J_0 \) of \( J \) such that \( \bigcup_{j \in J_0} C_{\alpha\beta}(G_j) = \tilde{1}. \)

Definition 4.5. \( (X,T) \) is called (α,β)-vague fuzzy strongly S-closed iff for every family \( \{G_j : T^*(G_j) \geq (\alpha, \beta), j \in J\} \) such that \( \bigcup_{j \in J} G_j = \tilde{1}, \) where \( \alpha \in I_0, \beta \in I_1 \) with \( \alpha + \beta \leq 1 \), there exists a finite subset \( J_0 \) of \( J \) such that \( \bigcup_{j \in J_0} G_j = \tilde{1}. \)

Definition 4.6. \( (X,T) \) is called (α,β)-vague fuzzy S-closed iff for every family \( \{G_j : G_j \subseteq (\alpha, \beta), j \in J\} \) such that \( \bigcup_{j \in J} G_j = \tilde{1}, \) \( \alpha \in I_0, \beta \in I_1 \) with \( \alpha + \beta \leq 1 \), there exists a finite subset \( J_0 \) of \( J \) such that \( \bigcup_{j \in J_0} G_j = \tilde{1}. \)

Proposition 4.11. Every (α,β)-vague fuzzy strongly S-closed space \( (X,T) \) is (α,β)-vague fuzzy S-closed.

Proposition 4.12. Let \( f : (X,T) \to (Y,S) \) be a (α,β)-vague fuzzy almost continuous onto mapping. If \( (X,T) \) is (α,β)-vague fuzzy strongly S-closed, then \( (Y,S) \) is (α,β)-vague fuzzy S-closed, where \( \alpha \in I_0, \beta \in I_1 \) with \( \alpha + \beta \leq 1 \).

5. Vague fuzzy T-spaces and its properties

Definition 5.1. \( (X,T) \) is called

(i) Vague fuzzy-T1 iff for A, B \( \subseteq \mathbb{C}^X \) with \( A \neq B \) there exist \( C, D \subseteq \mathbb{C}^X \) with \( T(C) \geq (\alpha, \beta), T(D) \geq (\alpha, \beta) \) such that either \( A \subseteq C, B \subseteq D \) or \( B \subseteq C, A \subseteq D \).

(ii) Vague fuzzy-T2 iff for A, B \( \subseteq \mathbb{C}^X \) with \( A \neq B \) there exist \( C, D \subseteq \mathbb{C}^X \) with \( T(C) \geq (\alpha, \beta), T(D) \geq (\alpha, \beta) \) such that \( A \subseteq C, B \subseteq D \) and \( C \neq D \).

Remark 5.1. From the above definition, it follows that vague fuzzy T2-space is a vague fuzzy T1-space, but the converse need not be true as shown in the example 5.1.

Example 5.1. Let \( X = \{a, b\} \) be a set. Let \( G_1 \) and \( G_2 \) be VFS’s of \( X \) defined as follows:
\( G_1 = \{a, 0.51, 0.44\}, \ (b, 0.3, 0.5), \ G_2 = \{a, 0.41, 0.65\}, \ (b, 0.3, 0.65)\).
We define VFT \( T : \mathbb{C}^X \to I \) as follows:
\[
T(a) = \begin{cases} 
\tilde{1} & \text{if } a = 0, \tilde{1}
\tilde{0} & \text{if } a = G_1
\tilde{0} & \text{otherwise},
\end{cases}
\]
Let \( \alpha = 0.4, \beta = 0.5 \). Let \( A_1, A_2 \subseteq \mathbb{C}^X \) be defined as follows:
\( A_1 = \{a, 0.45, 0.5\}, \ (b, 0.5, 0.45\}, \ A_2 = \{a, 0.4, 0.6\}, \ (b, 0.3, 0.65\} \). Clearly \( A_1 \neq A_2 \). Further for \( T(G_1) > (\alpha, \beta) \) and \( T(G_2) > (\alpha, \beta) \) we have \( A_1 \subseteq G_1, A_2 \subseteq G_2 \) and \( G_1 \neq G_2 \). This implies that \( (X,T) \) is not vague fuzzy-T2.
Definition 5.2.
Let \((X,T)\) be a vague smooth fuzzy topological space. For \(A, B \in \zeta^X\) and where \(\alpha \in I_0, \beta \in I_1\) with \(\alpha + \beta \leq 1\), \(A\) is called vague generalized fuzzy closed (vague gf-closed) if \(C_{\alpha\beta}(A) \subseteq B\) whenever \(A \subseteq B\) and \(T(B) \geq \langle \alpha, \beta \rangle\). The complement of a vague gf-closed set is a vague gf-open set.

Definition 5.3.
\((X,T)\) is called vague fuzzy \(T_{1/2}\), if every vague gf-closed set \(A \in \zeta^X\) is such that \(T^*(A) \geq \langle \alpha, \beta \rangle\).

Definition 5.4.
A mapping \(f : (X,T) \rightarrow (Y,S)\) is called vague gf-continuous if \(f^{-1}(A)\) is vague gf-closed for every \(S^*(A) \geq \langle \alpha, \beta \rangle\), \(A \in \zeta^Y\) and \(\alpha \in I_0, \beta \in I_1\).

Proposition 5.1. Let \(f : (X,T) \rightarrow (Y,S)\) and \(g : (Y,S) \rightarrow (Z,R)\) be vague gf-continuous mappings and let \((Y,S)\) be a vague fuzzy \(T_{1/2}\) space. Then \(gof\) is vague gf-continuous.

6. Vague fuzzy normal space

Definition 6.1.
\((X,T)\) is vague fuzzy normal if for every \(T^*(A) \geq \langle \alpha, \beta \rangle\), \(T(B) \geq \langle \alpha, \beta \rangle\) such that \(1 - A \subseteq B\), there exists \(C \in \zeta^X\) such that \(1 - A \subseteq I_0\beta(C) \subseteq C_{\alpha\beta}(C) \subseteq B\).

Proposition 6.1. In \((X,T)\) the following statements are equivalent.
(a) \((X,T)\) is vague fuzzy normal.
(b) For every \(T^*(A) \geq \langle \alpha, \beta \rangle, T(B) \geq \langle \alpha, \beta \rangle\) such that \(1 - A \subseteq B\), there exists \(C \in \zeta^X\) with \(T(C) \geq \langle \alpha, \beta \rangle\) such that \(1 - A \subseteq C \subseteq C_{\alpha\beta}(C) \subseteq B\).

References