Oscillation for Neutral Delay Differential Equations with “MAXIMA”*

*Ganesan V
Department of Mathematics, Aringar Anna Government Arts College, Namakkal -637002, Tamilnadu, India

Abstract

In this paper, some oscillatory results for neutral differential equations with ”Maxima” are obtained. Recently, considered Delay differential equations with ’maxima’ of the form

\[ (a(t)(x(t)\alpha') + q(t) \max_{t \in [t-\tau, t]} x_{\alpha}(r) = 0, \quad t \geq 0. \] (1)

We obtain some sufficient conditions for all solutions to be oscillatory and for all non oscillatory solution to be asymptotic.

Keywords and phrases: Oscillation; neutral differential equation; Asymptotic behavior, Maxima

2000 Mathematics Subject Classification. 34k15, 34k99, 34K11.

Main Results

Theorem 1.1. Assume that \(0 < \alpha < 1\). then every solution of (1) oscillation if, and only if

\[ \int_{t=0}^{t=\infty} \frac{q(t)}{a(t)} dt = \infty \] (2)

Theorem 1.2. Assume that \(\alpha > 1\). Suppose further that there exists \(\delta > \tau^{-1} I_{\alpha} \) Such that

\[ \lim_{t \to \infty} \inf_{t \in [t-\tau-\sigma, t]} \max_{t \in [t, t+\sigma]} \exp(-e^{\mu(t)}) > 0 \]

then every solutions of (1) oscillates.

In this paper we will consider the neutral differential equations with maxima y the form

\[ (a(t)(x(t) - p(t)x(t-\sigma) \alpha') + q(t) \max_{r \in [t-\tau, t]} x_{\alpha}(r) = 0, \quad t \geq 0 \] (3)

and

\[ (a(t)(x(t) + p(t)x(t-\sigma) \alpha') + q(t) \max_{r \in [t-\tau, t]} x_{\alpha}(r) = 0, \quad t \geq 0 \] (4)

where \(\alpha, \tau > 0\) and \(q(t)\) are defined as before, \(\sigma > 0\) and \(0 \leq p(t) < 1\). It is clear that equation (1) is a particular care of (3). We will discuss the oscillation of (3) and (4) in two case where \(\alpha < 1\) and \(\alpha > 1\). In the sequel for convenience, when we write a function in equalities without specifying its domains of validity, we assume mat it leads for all sufficiently large t.

Author Correspondence
Department of Mathematics
Aringar Anna Government Arts College, Namakkal 637002, Tamilnadu, India
E-mail address: ganesan_vgp@rediffmail.com
Theorem 1.3. Assume that \(0 \leq p(t) < 1\) and \(0 < \alpha < 1\). Then every solution (3) oscillates if and only if (2) holds.

Proof: The fact that oscillations of (3) implies of (2) can be found in [10,11,12,13]. Next, assume that \(x(t)\) is an eventually positive solution of (3). In view of Lemma 1 in [4,5,6,7], we get that

\[
z(t) = x(t) - p(t)x(t - \tau) \text{ is eventually positive.}
\]

Thus, \(x(t - \tau) = z(t - \tau) + p(t)x(t - \sigma - \tau) \geq z(t - \tau) + p(t)z(t - \sigma - \tau) \geq (1 + p(t))z(t)\).

Substituting it into (3), we have

\[
z'(t) + q(t)(1 + p(t))^{\alpha} z^{\alpha}(t) \leq 0.
\]

Thus,

\[
z^{-\alpha}(t) z'(t) + q(t)(1 + p(t))^{\alpha} \leq 0.
\]

We define \(m(t) = z(s) + (t - s) z'(s), \quad s \leq 1 \leq s + 1\). Since \(z'(s) \leq 0\), then \(z(s + 1) \leq m(t) \leq z(s)\) and

\[
\frac{m'(t)}{m^{\alpha}(t)} \leq \frac{z'(t)}{z^{\alpha}(t)} \quad (5)
\]

In view of (5),(6) and (2), we obtained

\[
\int_{m(N)}^{m(\infty)} \frac{dm}{m^{\alpha}} = -\infty \quad (6)
\]

which contradicts the facts \(\alpha \in (0, 1)\). The proof is complete.

If \(\alpha > 1, 0 \leq p(t) < 1\) and \(x(t)\) is an eventually positive solution of (3), then we have

\[
x(t - \tau) = z(t - \tau) + p(t)x(t - \sigma - \tau) = z(t - \tau) + p(t)z(t - \sigma - \tau) + p^{2}(t)x(t - \sigma - 2\tau) = z(t - \tau) + p(t)z(t - \sigma - \tau) + \cdots + p^{s}(t)x(t - \sigma - s\tau) + p^{s+1}(t)x(t - \sigma - (s + 1)\tau) \leq p^{t}(t)z(t - \sigma - s\tau)
\]

by (3), we have

\[
z'(t) + q(t) \max_{r \in [t - \tau, t]} p^{\alpha}(t)z(t - \sigma - s\tau) \leq 0.
\]

For any \(s \geq 0\). Note mat when \(p(t) > 0\),

\[
\lim_{t \to \infty} \inf_{t} q(t) \max_{r \in [t - \tau, t]} (-e^{N(t)}) > 0 \iff \lim_{t \to \infty} \inf_{t} p^{\alpha}(t)q(t) \max_{r \in [t - \tau, t]} (-e^{N(t)}) > 0.
\]

In view of theorem 2. We have the following results.

Theorem 1.4. Assume that \(\alpha > 1\) and \(0 \leq p(t) < 1\). Suppose further that for some non negative constants \(r\) there exists a \(\delta > (\sigma + s\lambda)^{-1} I_{m}\) (where \(s = 0\) if \(p(t) = 0\)) such that
\[
\lim_{t \to \infty} \inf q(t) \max_{r \in [t-\tau,t]} x^\alpha(r) (-e^{N(t)}) > 0
\]  

(7)

then every solution of (3) oscillates.

To obtain oscillatory results of equations (4). We need the following lemma which can be found in [3].

**Lemma 1.5.** An eventually concave function \(x(t)\) (i.e.; \(a(t)x''(t) \leq 0\) for all large \(t\)) is a constant sign eventually. If \(a(t)x(t) > 0\) and \((a(t)x(t)') \leq 0\) eventually and \((a(t)x(t)'')\) has a negative functions, then \((a(t)x(t)')\) is eventually positive. Furthermore, there is a member \(\mu \in (0, 1)\) such that \(a(t)x(t) \geq \mu(t)(a(t)x(t))'\) for all large \(t\).

**Proof.** Assume that \(x(t)\) is an eventually positive solutions of equation (4). Clearly, we have

\[
y(t) = a(t)x(t) + p(t)x(t-\tau) > \mu(t)(a(t)y(t))' > 0 \text{ and } (a(t)y(t))' \leq 0.
\]

Thus

\[
(a(t)x(t-\tau) = a(t)y(t-\tau) - p(t)y(t-\tau-\sigma) + p^2(t)x(t-\sigma-2\tau)
\]

\[
\geq (1 - p(t))\mu(t-\tau-\sigma) \geq (1 - p(t))\mu(t-\tau-\sigma)(a(t)y(t-\tau-\sigma))'
\]

substituting it into (4), we have

\[
((a(t)y(t))') + q(t) \max_{r \in [t-\tau,t]} (1 - p(t))\mu(t-\tau-\sigma)\alpha(a(t)y(t-\tau-\sigma))' \leq 0
\]

In view of theorem 1. We have the following theorem.

**Theorem 1.6.** Assume that \(a(0, 1)\) and \(p(t) \in [0, 1]\). Suppose further that

\[
\int_{t=0}^{\infty} q(t) \max_{r \in [t-\sigma,t]} x^\alpha(r) dt = \infty,
\]

then every solution of (4) oscillates. While if \(a > 1, p(t) \in [0, 1]\), and there exists \(\mu > (\lambda + \sigma)^{-1} I_{ms}\) such that

\[
\lim_{t \to \infty} \inf q(t) \max_{r \in [t-\sigma,t]} x^\alpha(r) \exp(-e^{\mu(t)}) > 0,
\]

then all solutions of (4) oscillates.

**References**


Received: January 10, 2015 | Revised: January 20, 2015 | Accepted: January 22, 2015 | Published: January 28, 2015

License: Ganesan V (2015). This is an open-access article distributed under the terms of the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original author and source are credited.